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MATHEMATICAL ANALYSIS

FUNCTIONS, LIMITS, SERIES,
CONTINUED FRACTIONS

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Functions, Limits, Series, Continued Fractions

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FOREWORD

THE PRESENT book, together with its companion volume devoted to the differential and integral calculus, contains the fundamental part of the material dealt with in the larger courses of mathematical analysis. Included in this volume are general problems of the theory of continuous functions of one and several variables (with the geometrical basis of this theory), the theory of limiting values for sequences of numbers and vectors, and also the theory of numerical series and series of functions and other analogous infinite processes, in particular, infinite continued fractions.

Chapter I, "The arithmetical linear continuum and functions defined there" (authors: L. A. Lyusternik and Ye. K. Isakova), is devoted to real numbers, the arithmetical linear continuum, limiting values, and to functions of one variable. The material of this chapter is more or less that which is usually called the introduction to mathematical analysis.

Chapter II, "n-dimensional spaces and functions defined there" (L. A. Lyusternik), effects the transition from functions of one variable to functions of n variables, which, geometrically, corresponds to the transition from the arithmetical linear continuum to n-dimensional space E_n, the fundamental theory of which is given. §1 is devoted to the fundamentals of n-dimensional geometry and, in particular, of the theory of orthogonal systems of vectors in E_n, which serves as a simpler model for the theory (Chapter IV) or orthogonal systems of functions. §2 is devoted to limiting values in E_n, to continuous functions of n variables and their systems (transformations in E_n). In this chapter also §3 deals with a subject which plays an important part in pure and applied mathematics, the theory of n-dimensional convex bodies.

Chapter III, "Series" (authors G. S. Salekhov and V. L. Danilov), consists of the theory of series and practical methods of summation.

The theory of numerical series is dealt with in §1 including questions relating to infinite products, double series and the summation of convergent series. Side by side with the classical material the reader will find new results about the general tests for the convergence of series and estimations of the remainder.

The more important classes of series of functions are considered in §2: power, trigonometrical, and also asymptotic power series, and their convergence. At this point some methods for the general summation of divergent series are added. In §3 are to be found various devices useful in calculations connected with the theory of series.

Chapter IV "Orthogonal series and orthogonal systems" (authors A. N. Ivanova and L. A. Lyusternik), contains the general problems of the reduction of functions to orthogonal (and also biorthogonal) series. Here, also, general orthogonal systems of polynomials and the classical systems of Legendre, Chebyshev, Hermite Polynomials, and others, are considered.

Chapter V "Continued fractions" (author A. N. Khovanskii), deals with that branch of analysis which occupied the attention of the greatest mathematicians of the eighteenth and nineteenth centuries, but which was afterwards unjustly forgotten. Continued fractions did not find a place in the contemporary larger courses of analysis; on the other hand comparatively recently, some elements of the theory of continued fractions were studied even in middle school. In the past few years the interest in continued fractions has revived in connection with their application in computation and other topics in applied mathematics.

Chapter VI, "Some special constants and functions" (authors L. A. Lyusternik, L. Ya. Tslaf and A. R. Yanpol'skii), has more of the nature of a manual (in the narrow sense of the word). The material here concerns various constants, the most important systems of numbers, including Bernoulli and Euler numbers, some discontinuous functions, and the simpler special functions (elliptic integrals, integral functions, the gamma and beta functions, some Bessel functions, etc.). These functions, together with orthogonal polynomials, after the elementary ones, are the most widely used in applications of mathematics. We would like to mention that these special functions will be dealt with more fully and in the complex domain in one of the following issues.

CHAPTER I

THE ARITHMETICAL LINEAR CONTINUUM AND FUNCTIONS DEFINED THERE

§ 1. Real numbers and their representation

1. Real numbers

All *real* numbers can be split into two classes: *rational* and *irrational.* All *integers* and *fractions (positive, negative* and *zero)* are rational numbers, while the remaining numbers are irrational.

The set of all rational numbers is *everywhere dense*, i.e. between any two distinct rational numbers a and b $(a<b)$ there is at least one further rational number c $(a<c<b)$, i.e. in fact, an infinity of rational numbers.

Examples of irrational numbers are: $\sqrt{2} = 1{\cdot}41421356\ldots$, $\pi = 3{\cdot}14159\ldots$, $e = 2{\cdot}7182818\ldots$ — the *base of natural logarithms*, and so on. Irrational numbers consist of *algebraic* and *transcendental* numbers. *Algebraic irrational* numbers are defined as all non-integral real roots of the algebraic equation

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0,$$

where a_i $(i = 1, 2, \ldots, n)$ are integers; for example, the roots $x = \sqrt[3]{10}$, $x = \sqrt[5]{8}$ of the equations $x^3 - 10 = 0$, $x^5 - 8 = 0$, the roots of the equation $x^5 - 3x^4 - 2x^3 + x^2 + 1 = 0$, and so on. The remaining irrational numbers are described as *transcendental*; examples of these are $\pi, e, e^\pi, 2^{\sqrt{2}}$, $\log n$ (where n is any integer not equal to 10^k) and so on.

2. The numerical straight line

We choose on a straight line E_1, an *origin* of measurement — the origin of coordinates 0, a *scale* the unit of length, and a direction — the orientation. We associate with every real number x a point $A(x)$ on the line E_1, having the coordinate (*abscissa*). x, conversely, with every point $A(x)$ on E_1, we associate a real number x — its abscissa. E_1 is called a *numerical straight line* or a *one-dimensional coordinate space* (see Chapter II for n-dimensional coordinate spaces E_n).

The number

$$|a| = \begin{cases} -a & \text{for } a < 0, \\ a & \text{for } a \geqq 0 \end{cases}$$

is called the *absolute* or *numerical* value of the number a.

The following relationships hold:

$$|a+b| \leqq |a|+|b|,$$
$$|a-b| \geqq ||a|-|b||,$$
$$|a\,.\,b| = |a|.|b|,$$
$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

The number $[a-b]$ is called the *distance* between the points a and b on the straight line E_1 (see Chapter II, § 1, sec. 1).

3. p-adic systems

Every real number is expressible as a decimal fraction, i.e. has a definite expansion in the decimal system of numeration. The decimal system is a particular case of a positional system to the base p, where any positive integer $p > 1$ can be taken as the base. The numbers $0, 1, 2, \ldots, p-1$ are called the *digits* of this system, while p^k $(k = 0, \pm 1, \pm 2, \ldots)$ are *units* of the kth order in the system.

Every positive integer N is *uniquely* expressible in the form

$$N = a_0p^0+a_1p^1+\ldots+a_np^n = \sum_{i=0}^{n} a_ip^i, \tag{1.1}$$

where a_i are digits. Equation (1.1) is written as

$$N = a_na_{n-1}a_{n-2}\ldots a_1a_0. \tag{1.1'}$$

Similarly, any positive real number S, rational or irrational, is expressible as a fraction to the base p:

$$S = \sum_{k=-\infty}^{n} a_k p^k, \tag{1.2}$$

which is written as

$$S = a_n a_{n-1} \ldots a_1 a_0, a_{-1} a_{-2} a_{-3} \ldots . \tag{1.2'}$$

If S is an irrational number, it is uniquely expressible by an infinite non-periodic fraction to the base p of the form (1.2) (or (1.2′)).

If S is rational, it is expressible as an infinite periodic fraction to the base p, e.g. the number $S = 1/6$ is written in the decimal system as

$$S = 0{\cdot}1666 \ldots = 0{\cdot}1\,(6).$$

In the binary system, $S = 1/6$ is expressed by the infinite fraction

$$S = 0{\cdot}0010101 \ldots = 0{\cdot}0\,(01) = \frac{1}{8} + \frac{1}{32} + \ldots .$$

Rational numbers to the base p are numbers expressible as fractions with denominator p^k ($k = 1, 2, -2 \ldots$); each such number has two forms in the system to the base p: one with 0 repeated, the other with $p-1$ repeated.

For example, the number $S = \frac{1}{2}$ is written in the binary system as

$$S = 0{\cdot}1000 \ldots = 0{\cdot}1\,(0),$$
$$S = 0{\cdot}0111 \ldots = 0{\cdot}0(1);$$

and in the decimal system as

$$S = 0{\cdot}5000 \ldots = 0{\cdot}5\,(0),$$
$$S = 0{\cdot}4999 \ldots = 0{\cdot}4\,(9).$$

Having selected one of these forms for rational numbers to the base p, say the first — with 0 repeated, we obtain a unique form for rational numbers to the base p as infinite periodic fractions to the base p, and at the same time a unique form for every real number.

The elements of various p-based systems were to be found in antiquity in different nations, and traces have been preserved into modern times in certain languages, e.g. of $p = 12$ (dozens and grosses), $p = 20$ (traces of this system have been preserved in French), $p = 40$ ("forty times forty" refers to number of churches in Moscow (*Trans.*)), etc. The system to the base 60 was well developed; it originated in ancient Babylon (traces are retained in measurement of angles and time). The 60-base system must have competed with the decimal in the Middle Ages in the Near East and Central Asia. The decimal system originated in India, was further developed in Central Asia, and passed from there into Europe.

At the present time the binary system is widely used in computers (together with the related systems having powers of two as base: $p = 2k$, $k > 1$ an integer). A system of numeration to the base three is used in the Moscow State University "Setun" computer. A set of numbers different from $0, 1, 2, \ldots, p-1$ is occasionally used for the p digits of the system to the base p. For instance, a convenient choice of digits in the system to the base 3 is $-1, 0, 1$. The digits -1 and 1 can be used in the binary system.

Non-homogeneous positional systems are more general; here, the ratios of units of different orders are different numbers. Such systems were used (before the introduction of the metric system) for representing "denominate" numbers, i.e. for representing magnitudes such as length, weight, etc. For example, the following system of units was used for measuring weight in pre-revolutionary Russia: 1 pud (≈ 16 kg) $=$ 40 funtov, 1 funt (≈ 400 g) $=$ 32 lota, etc.

4. Sets of real numbers

We shall discuss various sets of real numbers — for example, the set of *natural numbers:* $1, 2, 3, 4, \ldots, n, \ldots$, the set of all *proper fractions*, the set of all *rational numbers*, the set of all *real numbers* between 0 and 1, etc.

The numbers are called the *elements* of the set in question.

One can consider sets of elements of any kind, and not merely sets of real numbers. For instance, the set of points of a plane, the set of trees in a district, etc. The elements of these sets are respectively points of a plane, trees, etc.

Sets are denoted in this book by capital letters: M, N, A, B, X, Y, etc., or by the symbol $\{x_n\}$, where x_n are the elements of the set (*countable* sets). The set of numbers satisfying the inequalities $a < x < b$ (a, b are numbers) is called an *interval* and is written as (a, b). The sets of numbers satisfying the inequalities $x < a$, $x > b$, are called *infinite intervals* and are written as $(-\infty, a)$ and $(b, +\infty)$ respectively. The set of numbers x satisfying the inequalities $a \leqq x \leqq b$ is called a *segment* (or *closed interval*), and is written as $[a, b]$.

The sets of points x satisfying the inequalities

$$a \leqq x < b, \quad a < x \leqq b,$$

are called *semi-intervals* and are written respectively as $[a, b)$, $(a, b]$.

The *infinite semi-intervals* $(-\infty, a]$ and $[b, +\infty)$ are similarly defined.

The interval $(x-\varepsilon, x+\varepsilon)$, $(\varepsilon > 0)$ is called an *ε-neighbourhood* of the point x.

If an element x *belongs* (*does not belong*) to the set X, this is written symbolically as $x \in X$ ($x \overline{\in} X$ or $x \notin X$).

If all the elements of a set X are simultaneously elements of a set Y, X is said to be a *subset* of the set Y, and we write symbolically: $X \subset Y$. Otherwise, X is not a subset of Y, and we write this symbolically as $X \overline{\subset} Y$ (or $X \not\subset Y$). For example,

$$\frac{1}{3} \in (0, 1), \quad a \in [a, b), \quad a \overline{\in} (a, b), \quad b \overline{\in} [a, b);$$

$$(0, 1) \subset [0, 1), \quad [1, 2] \overline{\subset} (0, 1), \quad (a, b) \subset [a, b).$$

The set M of all the elements that belong both to a set A and a set B is called the *intersection* or *product* of the sets A and B, and is written symbolically as

$$M = A \cap B \quad (M = A \times B = A.B = AB).$$

For example,

$$(0, 1] = \left[-\frac{1}{2}, 1\right] \cap (0, 2), \quad b = (a, b] \cap [b, c)$$

and so on.

The set M consisting of all the elements that belong either to a set A or to a set B is called the *union* or *sum* of sets A and B, and is denoted symbolically by

$$M = A \cup B \ (M = A+B).$$

For example,

$$(0, 2) \cup \left[\frac{1}{2}, +\infty\right) = (0, +\infty), \ (-3, 7] \cup (5, 8] = (-3, 8].$$

The set M consisting of the elements of a set B that do not belong to a set A is called the *complement* of the set A with respect to the set B or the *difference* between the sets A and B, and is written symbolically as

$$M = B \setminus A \qquad (M = B - A).$$

For example,

$$(7,\ 8] = (5,\ 8] \setminus (-3,\ 7], \quad (0,\ 2) \setminus \left[0,\ \frac{1}{2}\right) = \left[\frac{1}{2},\ 2\right) \text{ etc.}$$

The notation $B \setminus A$ is employed in more general cases.

5. Bounded sets, upper and lower bounds

A set X of real numbers is said to be *bounded from above* (*below*) if there exists a number M (m), not less (not greater) than all the numbers $x \in X$. The number M (m) is called an *upper* (*lower*) *bound* of the set X.

A set X is said to be *bounded* if it is bounded from above and below. For example, the set $(-\infty, 0)$ is bounded from above, the set $(0, +\infty)$ bounded from below, while $(0, 1)$ is a bounded set.

The least (greatest) of all the upper (lower) bounds of a set X is called the *strict upper* (*lower*) *bound* M^* (m^*) of the set X and is written symbolically as

$$M^* = \sup_{x \in X} x \qquad \left(m^* = \inf_{x \in X} x\right).$$

The numbers M^* and m^* possess the following properties:

(1) The inequalities hold for all $x \in X$:

$$M^* \geqq x, \quad m^* \leqq x.$$

(2) Whatever the number $\varepsilon > 0$, a number $x_0 \in X$ can be found for which, respectively,

$$x_0 \geqq M^* - \varepsilon, \quad x_0 \leqq m^* + \varepsilon.$$

For example,

$$\sup_{x \in (-\infty,\, 0)} x = 0, \qquad \inf_{x \in (0,\, x)} x = 0.$$

(3) If the set $X = \{x\}$ is bounded from above (below), it has a strict upper (lower) bound.

The concepts introduced below (§ 2, sec. 2) of strict upper and lower bounds of a function are particular cases of the strict upper and lower bounds of a set.

6. The theory of irrational numbers

Precise irrational number theories, due to Dedekind, Cantor and Weierstrass, made their appearance in the second half of the nineteenth century, in connection with the critical consideration of the fundamental concepts of analysis.

Dedekind's theory.

The set of all rational numbers with all their properties is assumed given. The set of all rational numbers is divided into two classes A and A'. This division is called a *section in the rational number domain* if the following conditions are satisfied:

(*a*) every rational number falls into one and only one of the sets A and A'.

(*b*) every number a of the set A is less than every number a' of the set A'.

The set A' is called the *upper class*, and the set A *lower;* a section is denoted by $A|A'$.

Sections can be of three types:

(1) either there is no greatest number in the lower class A, while there is a least number r in the upper class A';

(2) or there is a greatest number r in the lower class A, while there is no least in the upper class A';

(3) or there is no greatest in the lower class, and no least in the upper class.

We say in the first two cases that the section *is performed by the rational number r* (which is the boundary between the classes A and A'), or that the section *defines the rational number r*. In the third case the section $A|A'$ defines no rational number; we say that a section of type (3) *defines some irrational number* α.

For example, if all the numbers $a \leqq 0$ are referred to class A, as also the numbers $a > 0$ for which $a^2 < 2$, while all the remainder are in class A', the section $A | A'$ defines the irrational number $\sqrt{2}$.

All real numbers can be ordered in the following way: two irrational numbers α and β, defined by the sections $A \mid A'$ and $B \mid B'$ respectively, are reckoned equal if the sections $A \mid A'$ and $B \mid B'$ are identical, and conversely, if the sections $A \mid A'$ and $B \mid B'$ coincide, the respective irrational numbers are said to be equal. We say that the number $\alpha > \beta$ if the class A wholly contains the class B, which does not coincide with A, and $\alpha > r$, where r is any rational number of the class A. Hence only one of the following relationships is possible for any two real numbers α and $\beta : \alpha = \beta, \alpha < \beta, \alpha > \beta$.

If sections are performed as described above in the domain of real numbers, it turns out that there always exists for any such section $A \mid A'$ a real number accomplishing the section. This is the essence of Dedekind's basic theorem. This property of the set of all real numbers is described by saying that it is *complete* or *continuous*.

We can introduce for real numbers the concepts of the arithmetical operations and laws (addition, multiplication, division by a non-zero number, etc.). For instance, the *sum* of two real numbers α and β is taken to be the real number $\gamma = \alpha + \beta$ which satisfies the relationship $a + b < \gamma < a' + b'$, where a, a', b and b' are all possible rational numbers satisfying the inequalities: $a < \alpha < a'$, $b < \beta < b'$.

All the other arithmetical operations can be similarly introduced, while retaining the fundamental properties.

Cantor's theory. We take all possible *fundamental sequences* (see § 3, sec. 2) of rational numbers. A sequence of rational numbers, convergent to a rational limit, is fundamental. At the same time, there exist fundamental sequences of rational numbers which do not have rational limits, as for instance the sequence of decimal approximations $\{1;\ 1{\cdot}4;\ 1{\cdot}41; \ldots\}$ of the square root of two.

Two infinite sequences $\{x_n\}$ and $\{y_n\}$ are said to be *equivalent* or *confinal* if $|x_n - y_n|$ tends to zero as $n \to \infty$. This means that two equivalent fundamental sequences $\{x_n\}$ and $\{y_n\}$ can only have the same rational limit x as $n \to \infty$. All equivalent fundamental sequences of rational numbers are referred to one class — the *equivalence class*, and the set of all fundamental sequences of rational numbers is divided into equivalence classes.

There are two possibilities: either there exists a rational number r, the common limit as $n \to \infty$ of all sequences $\{x_n\}$ of the same equivalence class X, or there is no such number among all the rational numbers.

We say in the first case that the equivalence class X *defines the rational number* r; in the second case, we say that the equivalence class X *defines an irrational number* x (which is also regarded as the limit of the sequences of the class X as $n \to \infty$). Every equivalence class defines a *real* (rational or irrational) *number.*

Arithmetical operations may also be introduced for all real numbers. For example, the sum $x+y$ of two real numbers x and y is taken to be the number which is defined by the class $X+Y$, where X is the equivalence class defining the number x, and Y the equivalence class defining the number y, the sum $X+Y$ being understood as the class to which sequences of the form $\{x_n+y_n\}$ belong, where $\{x_n\}$ is any sequence of X, and $\{y_n\}$ any sequence of Y.

All the other arithmetical operations on real numbers may be similarly introduced.

A real number x is said to be *positive* $(x>0)$, if the corresponding equivalence class X contains a fundamental sequence of positive rational numbers not convergent to zero.

The inequality $\alpha>\beta$, where α and β are two real numbers, means that $\alpha-\beta>0$.

The concepts of *fundamental sequence* and *equivalence class* can also be defined on the set E_1 of all real numbers. It turns out that all fundamental sequences are convergent on E_1 (see § 3, sec. 2), so that any equivalence class on E_1 defines a real number – the common limit of sequences convergent to it. No new numbers can be obtained with such a completion of the set E_1; in this sense, the set E_1 is *complete.*

Thus the real number set E_1 is obtained as a result of completion of the set of rational numbers by the limits of all possible fundamental sequences of rational numbers. This idea of completion has acquired great value in functional analysis.

As regards other irrational number theories, we may mention, in addition to Weierstrass's, A.N. Kolmogorov's argument (see ref. 6, p. 269) and the axiomatic construction of the real numbers (see ref. 4, p. 157, and ref. 15, p. 180).

§ 2. Functions. Sequences

1. Functions of one variable

If we are given a set of real numbers $X = \{x\}$, and each number $x \in X$ is associated with a corresponding number y, where $Y = \{y\}$ is the set of all such y, the *function* $y = f(x)$ is said to be *defined on the set* X. The set X is called the *domain of definition* of the function $f(x)$, and every number $x \in X$ the *argument*. The set Y is called the *range* of the function $f(x)$. If the argument x is given, the value $y = f(x)$ of the function is given. For instance, for the function $y = x^3$ the sets X and Y coincide with the real axis E_1; for the function $y = \tan x$ the set Y is the whole of the real axis E_1, while $X = E_1 - M$, where M is the set of all numbers of the form $\frac{1}{2}\pi + n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$; for the function $y = x!$ the sets Y and X are the sets of natural numbers; for the function $y = E(x)$ (the integral part of x), X is the real line E_1, and Y is the set of natural numbers. For the function

$$\operatorname{sign} x = \begin{cases} -1 & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x = 0, \\ 1 & \text{for} \quad x > 0 \end{cases}$$

the set X is the real line, while the set Y consists of three numbers: -1, 0, 1.

Any finite set of numbers $\{a_i\}$ $(i = 1, 2, \ldots, n)$ can be regarded as a function, given on the finite set of natural numbers $X < \{1, 2, \ldots, n\}$ and associating with each of these numbers i the value of the function $f(i) = a_i$ $(i = 1, 2, \ldots, n)$.

The concept of a function has been subjected to wide generalization. X can be a set of arbitrary elements. A *numerical function* is said to be given on this set if a number $f(x)$ is associated with every element x of the set.

We shall consider in Chapter II functions defined on a set of points (or vectors) of n-dimensional space. The area bounded by a polygon, or its perimeter, can be regarded as functions defined on a set of plane polygons; physical magnitudes, such as the mass of a body, its charge, etc. are defined on the set of corresponding physical bodies, etc.

The elements of a set X, on which a function is defined, are occasionally called *points.*

2. Upper and lower bounds of a function

An *upper* (*lower*) bound of a function $f(x)$, defined on a set X, is a number $M(m)$ such that $f(x) \leqq M(f(x) \geqq m)$ for all $x \in X$. If this number exists, the function $f(x)$ is said to be *bounded from above* (*below*) on X. A function, bounded from above and below on X, is said to be *bounded on X.*

The least (greatest) of all the upper (lower) bounds of a function $f(x)$ is called the *strict upper* (*lower*) *bound* M^* (m^*) of the function and is written as

$$M^* = \sup_{x \in X} f(x) \qquad \left(m^* = \inf_{x \in X} f(x)\right).$$

If there exists an element x_0 (x_1) of X, for which

$$f(x_0) = \sup_{x \in X} f(x) \qquad \left(f(x_1) = \inf_{x \in X} f(x)\right),$$

then

$$\sup_{x \in X} f(x) = f(x_0) \qquad \left(\inf_{x \in X} f(x) = f(x_1)\right)$$

is called the *absolute maximum* (*minimum*) *of the function* $f(x)$ and is written as

$$f(x_0) = \sup_{x \in X} f(x) \equiv \max_{x \in X} f(x) \qquad \left(f(x_1) = \inf_{x \in X} f(x) \equiv \min_{x \in X} f(x)\right).$$

In this case, we say that $f(x)$ *attains its absolute maximum* (*minimum*) *at the point* x_0 (x_1)

Given the finite set $a_1, a_2, \ldots, a_n$, we write

$$\max_n \{a_1, a_2, \ldots, a_n\} \qquad \left(\min_n \{a_1, a_2, \ldots, a_n\}\right)$$

for the *maximum* (*minimum*) of the numbers $a_1, a_2, \ldots, a_n$. For example,

$$\inf_{x \in (0, \infty)} \frac{1}{x} = 0, \quad \min_{x \in E_1} \sin x = -1, \quad \max_{x \in E_1} \sin x = 1,$$

$$\max\{4, 3, 7, 11, 8\} = 11, \quad \min\{4, 3, 2, 10, 17\} = 2.$$

The following inequalities hold:

(1)
$$\sup_{x\in X} f(x)+\sup_{x\in X} f_1(x) \geqq \sup_{x\in X} (f(x)+f_1(x)), \tag{1.3}$$
$$\inf_{x\in X} f(x)+\inf_{x\in X} f_1(x) \leqq \inf_{x\in X} (f(x)+f_1(x)). \tag{1.4}$$

If $f(x)$, $f_1(x)$ and $f_1(x)+f(x)$ attain a maximum (minimum) on X, then we have

$$\max_{x\in X} f(x)+\max_{x\in X} f_1(x) \leqq \max_{x\in X} (f(x)+f_1(x)), \tag{1.3a}$$
$$\min_{x\in X} f(x)+\min_{x\in X} f_1(x) \leqq \min_{x\in X} (f(x)+f_1(x)). \tag{1.4a}$$

The sign of equality holds in (1.3a) (or in (1.4a)) when $f(x)$ and $f_1(x)$ attain a maximum (minimum) at the same point.

For example,

$$\max_{x\in E_1} \sin x+\max_{x\in E_1} \cos x = 2 > \max_{x\in E_1} (\sin x+\cos x) = \sqrt{2}.$$

(2) If $Y\subset X$ (the set Y is part of the set X), we have

$$\sup_{y\in Y} f(x) \leqq \sup_{x\in X} f(x), \quad \inf_{y\in Y} f(y) \geqq \inf_{x\in X} f(x).$$

For example,

$$\max_{x\in\left[0,\,\frac{\pi}{4}\right]} \sin x = \frac{\sqrt{2}}{2} < \max_{x\in E_1} \sin x = 1,$$
$$\max(3,\ 11,\ 15,\ 8) = 15 > \max(3,\ 11,\ 8) = 11.$$

The following notation is often used,

$$(f(x) = a),\quad (f(x) < a),\quad (f(x) \leqq a),\quad (f(x) > a),\quad (f(x) \geqq a)$$

and similar ones, to denote the sets of points x for which the respective inequalities are satisfied (they are called Lebesgue sets).

For instance, $(x^2 < 2)$ is the interval $(-\sqrt{2},\ \sqrt{2})$; $(\sin x = 1)$ is the set of numbers $\left\{(4n+1)\frac{1}{2}\pi\right\}$, where n is any integer.

3. Even and odd functions

Let us consider functions defined on the real axis or on the segments $[-a, a]$ and intervals $(-a, a)$ (in general on a set M, symmetric with respect to the origin).

A function $f(x)$ is said to be *even* if $f(-x) = f(x)$ for any x of its domain of definition, and *odd* if $f(-x) = -f(x)$. All even powers x^{2n} are even functions, and all odd x^{2n+1} are odd functions. Other examples of even functions are $\cos x$, $|x|$, and of odd: $\sin x$, $\tan x$, etc.

A sum of even functions is even, and of odd, odd. The product of even functions is even; the product of an even number of odd functions is even, and of an odd number, odd. For example, $\sin x \tan x$ is an even function, $x \sin x \tan x$ is odd. The product (and quotient) of an odd and even function is odd. For example, $|x| \sin x$ is odd.

A constant is an even function.

Any function of an even function is an even function, e.g. $e^{|x|}$, $\sin(\cos x)$ are even functions. An even function of an odd function is an even function, e.g. $\cos(\sin x)$. An odd function of an odd function, is an odd function, e.g. $\tan(\sin x)$.

Any function $f(x)$ is expressible as the sum of an even function $f_1(x)$ and an odd function $f_2(x)$:

$$f(x) = f_1(x) + f_2(x),$$

where

$$f_1(x) = \frac{1}{2}[f(x) + f(-x)] + C,$$

$$f_2(x) = \frac{1}{2}[f(x) - f(-x)] - C$$

(C is a constant).

4. Inverse functions

Given two sets X and Y, each element $x \in X$ being associated with some element $y = A(x) \in Y$, a *mapping* (or *correspondence*) A of the set X into the set Y is said to be given.

If $y = A(x)$, $y \in Y$ is called the *image* of the element $x \in X$, while x is the *pre-image* of y.

If a unique element $x \in X$ corresponds as pre-image to each element $y \in Y$, the mapping (correspondence) A is called a *one-to-one mapping* of X into Y.

EXAMPLE 1. Let all the houses of a street be numbered by the integers from 1 to 80. We have a one-to-one correspondence between the set of houses and the set of the first 80 natural numbers.

EXAMPLE 2. Let E' be the set of all non-zero real numbers; one of the two signs $+$ or $-$ is associated with each number $x \in E'$. We have a mapping of E' into the set of signs consisting of two elements; the pre-images of the sign $+$ $(-)$ are all positive (negative) numbers.

Let the sets X and Y be sets of the numerical axis E_1; the mapping $A \equiv f(x)$ of the set X into Y is some function $y = f(x)$, defined on the set X, with the domain of values Y.

If the function $y = f(x)$ is a one-to-one mapping of the set X into the set Y, we say that there exists for the function $f(x)$ the *inverse function* $x = \varphi(y)$, which maps the set Y into the set X. The set Y is the domain of definition of the function $x = \varphi(y)$, while the set X is its range.

EXAMPLE 3. $y = \sin x$, $x = \operatorname{arc\,sin} y$, $X = \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$.

EXAMPLE 4. $y = \tan x$, $x = \operatorname{arc\,tan} y$, $X = \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$.

EXAMPLE 5. $y = e^{kx}$ $(k \neq 0)$, $x = (1/k)\log y$, $X = E_1$.

5. Periodic functions

A function $f(x)$ is called *periodic* if a number $\omega > 0$ exists such that, for any x,

$$f(x+\omega) = f(x). \tag{1.5}$$

The number ω is called a *period* of the function $f(x)$. If ω_1 and ω_2 are periods of $f(x)$, $\omega_1+\omega_2$ is also a period of $f(x)$.

The least of all such positive numbers ω is called the *least period* (or simply the *period*) of the function $f(x)$.

Theorem 1. *If a continuous* (see § 3, sec. 11) *function $f(x)$ is periodic and differs from a constant, then it has a least period $\omega_0 > 0$, and all its remaining periods are multiples of ω_0.*

For instance, when $f(x) = \sin x$ we have $\omega_0 = 2\pi$, when $f(x) = |\sin x|$ the period $\omega_0 = \pi$, and when $f(x) = E(x)$ (see § 2, sec 1) the period $\omega_0 = 1$.

6. Functional equations

A *functional equation* is an equation connecting different values of functions; we say that a function, for which such an equation holds, is a *solution* of the functional equation, or, that the functional equation is *satisfied by it.*

Example 6. Solutions of the functional equation

$$f(x+y) = f(x)+f(y) \tag{1.6}$$

include the linear functions

$$f(x) = kx \qquad (k \text{ is a constant}).$$

It can be shown that they are the only continuous functions satisfying this functional equation.

Example 7. Solutions of the functional equation

$$f(x)\cdot f(y) = f(x+y) \tag{1.7}$$

include the exponential functions $a^x (a \geqq 0)$, these being the only continuous functions that satisfy this equation.

Notice that periodic functions satisfy the functional equation

$$f(x+\omega) = f(x).$$

We can also discuss systems of functional equations. For instance, the pair of functions

$$f(x) = \sin x, \quad \varphi(y) = \cos y$$

(and the pair of functions $f(x) = 0$, $\varphi(x) = 0$) is the unique continuous solution of the system of functional equations

$$f(x+y) = f(x)\varphi(y)+f(y)\varphi(x),$$
$$\varphi(x+y) = \varphi(x)\varphi(y)-f(x)f(y),$$

if we required in addition that the functions be positive in the interval $\left(0, \frac{1}{2}\pi\right)$ and that the conditions $f\left(\frac{1}{2}\pi\right) = 1, \varphi\left(\frac{1}{2}\pi\right) = 0$ be satisfied.

7. Numerical sequences

A function, defined on the set of positive integers, is called an *infinite numerical sequence if for each positive integer* $n\,(n = 1, 2, 3, \ldots)$ *there is a corresponding number* x_n *— the n-th term of the sequence* $\{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\}$. The number n in the expression x_n is called the *index* (or *subscript*). We sometimes consider sequences $\{x_n\}$ where, in addition to the natural values, n can be zero or have any integral value.

We shall say that, when $m > n$, the term x_m *follows* the term x_n (x_n *precedes* x_m), independently of the actual magnitude of the numbers x_n and x_m. A sequence is regarded as given if a relation is known for forming the terms of the sequence. It is often possible to find an expression (formula) for the *general term* x_n of a sequence.

EXAMPLE 8. *Arithmetical progression*

$$x_n = a + (n-1)a.$$

EXAMPLE 9. *Geometrical progression*

$$x_n = aq^{n-1}.$$

EXAMPLE 10. Decimal approximations to the number $\pi = 3{\cdot}14159\ldots$:

$$x_1 = 3; \quad x_2 = 3{\cdot}1; \quad x_3 = 3{\cdot}14; \quad x_4 = 3{\cdot}141; \ \ldots .$$

EXAMPLE 11. Let $n = a_1 a_2 \ldots a_k$ be a given number in the decimal system, where $a_1, a_2, \ldots, a_k$ are digits; then the numbers $x_n = 0 \cdot a_1 a_2, \ldots a_k$ and $y_n = 0 \cdot a_k a_{k-1} \ldots a_1$ form numerical sequences when $n = 1, 2, \ldots$. For example, when

$$n = 15 \text{ we have } x_{15} = 0{\cdot}15; \ y_{15} = 0{\cdot}51.$$

EXAMPLE 12. We decompose the positive integer n into prime factors

$$n = p_1^{2l_1} \cdot p_2^{2l_2} \ldots p_k^{2l_k} \cdot p_{k+1}^{2l_{k+1}+1} \ldots p_m^{2l_m+1}$$

and define a sequence $\{x_n\}$ as follows:

$$x_n = p_1^{l_1} \cdot p_2^{l_2} \dots p_k^{l_k} \cdot p_{k+1}^{-(l_{k+1}+1)} \dots p_m^{-(l_m+1)}$$

with $n = 1, 2, 3, \dots$.

For example, $n = 18 = 2^1.3^2$, $x_{18} = 3.2^{-1} = 3/2$.

It is possible to discuss sequences of vectors, functions, etc., as well as numerical sequences.

8. Upper and lower bounds of a sequence

A sequence $X = \{x_n\}$ is said to be *bounded from above* by a number M if $x_n \leqq M$ for all indices n. In accordance with the definition given in § 1, sec. 5, the number M is called an *upper bound* of the sequence $\{x_n\}$. The sequence $X = \{x_n\}$ is said to be *bounded from below* by the number m, if $x_n \geqq m$ for all n. The number m is called a *lower bound* of the sequence $\{x_n\}$. The sequence is said to be *bounded* if it is bounded from above and below.

The least M^*(greatest m^*) of all the upper bounds M (lower bounds m) is called the *strict upper* (*lower*) *bound* of the sequence $\{x_n\}$ and is denoted, as before, by

$$\sup_n x_n = M^* \quad \text{and} \quad \inf_n x_n = m^*.$$

The propositions of § 2, sec. 2, hold for the strict upper and lower bounds of a sequence $\{x_n\}$.

9. Maximum term of a sequence

Given a sequence $\{u_k\}$ ($k = 0, 1, 2, \dots$), we sometimes want to find $\max\limits_k u_k$ – the *maximum term of the sequence* $\{u_k\}$, provided such a term exists; let us call it μ:

$$\mu = \max_k u_k.$$

The subscript of the maximum term of $\{u_k\}$ is called the *central subscript* and is denoted by ν. If the terms of $\{u_k\}$ include several that are equal to μ, the greatest of the subscripts of these terms is taken as ν.

If $u_n = u_n(x)$ are functions of x, $x \in X \subset E_1$, then $\nu = \nu(x)$, $\mu = \mu(x)$ are also functions of x.

Example 13. Given the sequence $\{u_k(x) = x^k/k!\}$ ($k = 0, 1, 2, \dots$), the number $\nu(x) = E(x) = n$, while the maximum term $\mu(x) = x^n/n!$ for non-integral x; if x is an integer, $x = m$, then $\nu(x) = m$, while $\mu(x) = u_{m-1} = u_m$. For example, with $x = 6$ we have $\nu(6) = 6$, $\mu(6) = u_5 = 6^5/5! = 6^6/6!$

EXAMPLE 14. Given the sequence $\{u_k(x) = x^k/(2k+1)!\}$ ($k = 0\ 1, 2, \ldots$), the index of the maximum term is defined by the inequality

$$2n(2n+1) \leqq x \leqq (2n+2)(2n+3),$$

and, with fixed $x > 0$, the maximum term u_n corresponds to the value $n = E(x)$.

10. Monotonic sequences

A sequence $\{x_n\} = \{x_1, x_2, \ldots, x_n\}$ is said to be *monotonically increasing (decreasing)*, if

$$\left.\begin{array}{c} x_1 < x_2 < x_3 < \ldots < x_n < \ldots \\ (x_1 > x_2 > x_3 > \ldots > x_n > \ldots), \end{array}\right\} \tag{1.8}$$

and to be *non-decreasing*(*non-increasing*), if

$$\left.\begin{array}{c} x_1 \leqq x_2 \leqq x_3 \leqq \ldots \leqq x_n \leqq \ldots \\ (x_1 \geqq x_2 \geqq \ldots \geqq x_n \geqq \ldots. \end{array}\right\} \tag{1.9}$$

Sequences of the form (1.8) and (1.9) are described as *monotonic*, sequences of the form (1.8) being *monotonic in the strict sense* (or *strictly monotonic*). For example, the sequences $\{x_n\}$, where

$$x_n = 2 - \frac{1}{2^{n/3} + \frac{1}{3}}, \quad x_n = 5 - \frac{1}{3^{n-1}}, \quad x_n = 2 + 5(n-1)$$

are monotonically increasing in the strict sense; the sequences $\{x_n\}$, where

$$x_n = \frac{1}{n}, \quad x_n = \frac{1}{\sqrt{n^2+1}},$$

are monotonically decreasing in the strict sense; the sequence $\{x_n\}$, where

$$x_1 = 1; \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{1}{2}, \quad x_4 = \frac{1}{4}, \quad x_5 = \frac{1}{4}, \ldots$$

is non-increasing; the sequence $\{x_n\}$, where

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = 2, \quad x_5 = 3, \quad x_6 = 4, \ldots$$

is non-decreasing.

All the above-mentioned sequences are monotonic.

11. Double sequences

A *double sequence* $\{a_{nm}\}$ is a function of a pair of integral subscripts n and m: to each pair of integers n and m there corresponds a number a_{nm} – the term of the sequence $\{a_{nm}\}$.

EXAMPLE 15. $\{a_{nm} = 1/(n^2+m^2)\}$ $(n, m = 1, 2, \ldots)$.

If the numbers n and m are represented in the decimal system by

$$\left.\begin{aligned} n &= a_k a_{k-1} \ldots a_1, \\ m &= b_k b_{k-1} \ldots b_1 \end{aligned}\right\} \tag{1.10}$$

(it can be assumed that n and m are represented by the same number of digits; this can always be arranged by putting a suitable number of zeros in front), we can form a double sequence $\{N_{nm}\}$, where

$$N \equiv N_{nm} = a_k b_k a_{k-1} b_{k-1} \ldots a_1 b_1. \tag{1.11}$$

EXAMPLE 16. If $n = 103$, $m = 27 = 027$, then $N_{103,27} = 100\,237$.

Conversely, we can associate with every integer N, represented by the right-hand side of (1.11) (if the number of digits $(2k-1)$ in N is odd we put the digit $a_{2k}= 0$ in front), a corresponding pair of integers n and m in accordance with (1.10).

We thus obtain a one-to-one correspondence between the set of pairs of integers and the set of all non-negative integers.

EXAMPLE 17. Given the double sequence $\{a_{nm}\}$ $(n, m = 0, 1, 2, \ldots)$, it can be related thus to an ordinary sequence $\{b_N\}$ $(N = 0, 1, 2, \ldots)$, when $N = N_{nm}$, we have $b_N = a_{nm}$.

The double sequence $\{a_{nm}\}$ is said to be *arranged* as the ordinary sequence $\{b_N\}$.

See Chapter III for the summation of double sequences (double series).

§ 3. Passage to the limit

1. The limit point of a set

We call x_0 a *limit point* of a set X if there is at least one more point of X (different from x_0) in any ε-neighbourhood of the point x_0 (ε is any positive number). Alternatively, x_0 is a *limit point* of a set

X if an infinity of points of X is contained in any ε-neighbourhood of x_0. A set may have no (or even infinitely many) limit points, which may or may not belong to the set. For instance, the set $X = \{1/n\}$, where $n \geqq 1$, has one limit point 0, which does not belong to the set; every point of any interval (a, b) (as also the points a and b) is a limit point of the interval.

THEOREM 2 (Bolzano–Weiestrass). *Every bounded infinite set X of E_1 has at least one limit point.*

A set M is said to be *closed* if it contains all its limit points.

For example, $M = \left\{0, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ is a closed set; any segment $[a, b]$ is a closed set.

A set M in E_1 is described as *open* or as a *domain* if any point appears in M along with its ε-neighbourhood.

For example, any interval (α, β) is an open set; any system of intervals is an open set.

The real axis is an open and a closed set; on the other hand, any semi-interval $(\alpha, \beta]$, $[\alpha, \beta)$ is neither an open nor a closed set.

The complement of an open (closed) set M with respect to E_1 is a closed (open) set.

For instance, the complement of the open set $(-\infty, 0) \cup (1, +\infty)$ is the closed set $[0, 1]$.

2. The limit point and limit of a sequence

The number x is called a *limit point of the sequence* $X = \{x_n\}$ if any ε-neighbourhood of x contains at least one term x_m of X different from x (i.e. an infinity of terms of the sequence).

A bounded infinite sequence $X = \{x_n\}$ has at least one limit point.

The constant number x_0 is called a *limit* of the sequence $\{x_n\}$ if, given any number $\varepsilon > 0$, however small, there is a subscript N (depending only on the choice of ε) such that, for all $n > N$:

$$|x_n - x_0| < \varepsilon.$$

We say in this case that $\{x_n\}$ *has a finite limit* x_0 and write $\lim\limits_{n\to\infty} x_n = x_0$, or that $\{x_n\}$ is *convergent* to x_0, and write $x_n \to x_0$. The sequence $\{x_n\}$ is said to be *convergent* in this case. This means geometrically that, given any $\varepsilon > 0$, from some subscript N_ε onwards, all the points x_n lie in the neighbourhood $(x_0-\varepsilon, x_0+\varepsilon)$. In particular, x_n is described as an *infinitesimal* if $\lim\limits_{n\to\infty} x_n = 0$. A sequence that has no limit is said to be *divergent.*

A sequence $\{x_n\}$, having the property that, given any $\varepsilon < 0$, there exists a subscript N such that $|x_m-x_n| < \varepsilon$ for all $m > N$, $n > N$, is described as *fundamental.*

A general criterion for the convergence of a sequence $\{x_n\}$ is:

THE BOLZANO–CAUCHY CRITERION. *A necessary and sufficient condition for the sequence $\{x_n\}$ to be convergent is that it be fundamental.*

We say that the limit of the sequence $\{x_n\}$ is equal to infinity,

$$\lim_{n\to\infty} n_n = \infty,$$

if, no matter how large the number $K > 0$, there exists N such that, for all $n > N$,

$$|x_n| > K.$$

If the numbers $x_n(n > N)$ are positive (negative), we write

$$\lim_{n\to\infty} x_n = +\infty \qquad \left(\lim_{n\to\infty} x_n = -\infty\right).$$

Some limits:

1. $$\lim_{n\to\infty} \frac{an^k+a_1n^{k-1}+\ \dots\ +a_n}{bn^k+b_1n^{k-1}+\ \dots\ +b_n} = \frac{a}{b} \qquad (b \neq 0).$$

2. $$\lim_{n\to\infty} \frac{(\sqrt[n]{a}+\sqrt[n]{b})^n}{2} = \sqrt{ab}.$$

3. $$\lim_{n\to\infty} \sqrt[n]{a} = 1 \quad \text{for} \quad a > 1.$$

4. $$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

5. $$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e \qquad \text{[cf. (6.36)].}$$

6. $$\lim_{n\to\infty} \frac{n}{\sqrt[n]{(n+1)(n+2)\dots 2n}} = \frac{e}{4}.$$

7. $$\lim_{n\to\infty} \frac{1+\sqrt{2}+\sqrt[3]{3}+\ \dots\ +\sqrt[n]{n}}{n} = 1.$$

8. For $p > 1$

$$\lim_{n\to\infty}\left\{\frac{n}{p-1}-\left[\left(\frac{n}{n+1}\right)^p+\left(\frac{n}{n+2}\right)^p+\ \dots\right]\right\}=\frac{1}{2}.$$

9. For $\alpha > 0$.

$$\lim_{n\to\infty}\frac{1^{\alpha-1}-2^{\alpha-1}+3^{\alpha-1}-4^{\alpha-1}+\ \dots\ +(-1)^{n-1}n^{\alpha-1}}{n^\alpha}=0.$$

10. If we write

$$A_n=\frac{a_0+a_1+\ \dots\ +a_n}{n+1}\quad\text{and}\quad G_{n-1}=\sqrt[n]{a_0a_1\ \dots\ a_{n-1}}\ ,$$

we have, with $a_k = C_n^k$:

$$\lim_{n\to\infty}\sqrt[n]{A_n}=2,\qquad \lim_{n\to\infty}\sqrt[n]{G_n}=\sqrt{e}.$$

11. If the above notation is retained for A_n and G_n, we have with $a_k = a+kd (k = 0, 1, 2, \dots), a > 0, d > 0$:

$$\lim_{n\to\infty}\frac{G_n}{A_n}=\frac{2}{e}.$$

3. Fundamental theorems concerning limits

1°. *A sequence can have only one limit.*

2°. *If a sequence has a finite (infinite) limit, it is bounded (non-bounded).*

3°. *If a sequence* $\{x_n\}$ *has a unique limit point* x_0, *the sequence is convergent and* x_0 *is its limit. Conversely, if a sequence* (x_n) *is convergent to* x_0, x_0 *is its unique limit point.*

4°. *If* x *is a limit point of a sequence* $\{a_n\}$, *there exists a subsequence* $\{a_{n_k}\}$, *convergent to* x. *Conversely, if* y *is the limit of some subsequence* $\{a_{n_k}\}$, y *is the limit point of the sequence* $\{a_n\}$.

5°. *Similarly, if* x *is a limit point of a set* M *of the numerical axis* E_1, M *contains a sequence* $\{x_n\}$ *of numbers different from* x *that converges to* x.

6°. *On the assumption that* $\lim_{n\to\infty} x_n$ *and* $\lim_{n\to\infty} y_n$ *exist, we have:*

$$\left.\begin{aligned}\lim_{n\to\infty}(x_n \pm y_n) &= \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n,\\ \lim_{n\to\infty}(x_n \cdot y_n) &= \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n,\\ \lim_{n\to\infty}\frac{x_n}{y_n} &= \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}, \text{ where } \lim_{n\to\infty} y_n \neq 0;\end{aligned}\right\}$$

if $x_n < y_n$, *then*

$$\lim_{n\to\infty} x_n \leqq \lim_{n\to\infty} y_n.$$

4. Some propositions on limits

1°. *If* $\lim\limits_{n\to\infty} a_n = a$ *and all the* $a_i > 0$ $(i = 1, 2, \ldots)$, *then*

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \ldots a_n} = a, \qquad \lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = a.$$

2°. *If* A *is the greatest of the numbers*

$$a_1, a_2, \ldots, a_n \, (a_i \geqq 0)$$

and

$$p_i > 0 \quad (i = 1, 2, 3, \ldots, n),$$

we have

$$\left.\begin{aligned}\lim_{m\to\infty} \sqrt[m]{p_1 a_1^m + p_2 a_2^m + \ldots + p_n a_n^m} &= A,\\ \lim_{m\to\infty} \frac{p_1 a_1^{m+1} + p_2 a_2^{m+1} + \ldots + p_n a_n^{m+1}}{p_1 a_1^m + p_2 a_2^m + \ldots + p_n a_n^m} &= A.\end{aligned}\right\}$$

3°. *If, for a sequence* $\{a_n\}$,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \bar{a}$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$$

both exist, then

$$\bar{a} = a.$$

4°. *If*

$$\lim_{n\to\infty}\frac{p_n}{p_0+p_1+\ldots+p_n}=0,\quad p_i>0,$$

and, as $n\to\infty$, s_n has a limit equal to s, we have

$$\lim_{n\to\infty}\frac{s_0p_n+s_1p_{n-1}+s_2p_{n-2}+\ldots+s_np_0}{p_0+p_1+p_2+\ldots+p_n}=s.$$

5. Upper and lower limits of a sequence

The *upper* (*lower*) *limit of a sequence* $\{x_n\}$ is the strict upper (lower) bound of the set numbers which are limits of the sequence, and is written as

$$\overline{\lim_{n\to\infty}}\,x_n\quad\left(\underline{\lim_{n\to\infty}}\,x_n\right).$$

For example,

$$\overline{\lim_{n\to\infty}}\left[(-1)^n+\frac{1}{n}\right]=1,$$

$$\underline{\lim_{n\to\infty}}\left[(-1)^n+\frac{1}{n}\right]=-1,$$

while at the same time,

$$\sup_{n=1,2,\ldots}\left[(-1)^n+\frac{1}{n}\right]=\frac{3}{2},$$

$$\inf_{n=1,2,\ldots}\left[(-1)^n+\frac{1}{n}\right]=-1.$$

Every bounded sequence has an upper and a lower limit.

If a sequence is convergent, its limit coincides with its upper and lower limits; if the upper and lower limits are the same, the sequence is convergent to their common value.

Given any sequence that has an upper and a lower limit, we can readily form monotonic sequences convergent respectively to the upper and lower limits: a monotonically non-increasing $\{x_{*n}\}$ to the lower, and a monotonically non-decreasing $\{x_n^*\}$ to the upper:

$$x_n^* = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\},$$

$$x_{*n} = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\},$$

$$\overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} x_n^*, \quad \underline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} x_{*n}.$$

If $\{x_n\}$ is a monotonically non-decreasing (non-increasing) sequence and $a = \lim\limits_{n\to\infty} x_n$, then $x_n^* = x_n$, $x_{*n} = a$ $(x_n^* = a,\ x_{*n} = x_n)$.

6. Uniformly distributed sequences

Let the sequence $\{x_n\}$ lie on the segment $[a, b]$.

Let $N_n(\alpha, \beta)$ denote the number of the points x_k $(k = 1, 2, \ldots, n)$ which lie in the interval $(\alpha, \beta) \subset [a, b]$. If the limit

$$\lim_{n\to\infty} \frac{N_n(\alpha, \beta)}{n} = \sigma = \beta - \alpha$$

exists, then no matter what the interval $(\alpha, \beta) \subset [a, b]$, the sequence is said to be *uniformly distributed on the segment* $[a, b]$.

EXAMPLE 18. The sequence $\{y_n\}$ of EXAMPLE 11 (see § 2, sec. 7) is uniformly distributed on $[0, 1]$.

EXAMPLE 19. The sequence $\{x_n\}$, where

$$x_n = an^\sigma - [an^\sigma], \quad a > 0,\ 0 < \sigma < 1 \qquad (n = 1, 2, \ldots),$$

($[x]$ is the integral part of x) is uniformly distributed on $[0, 1]$.

EXAMPLE 20. The sequence $\{x_n\}$, where

$$x_n = a(\ln n)^\sigma - [a(\ln n)^\sigma], \quad a > 0,\ \sigma > 1 \qquad (n = 1, 2, \ldots),$$

is uniformly distributed on the segment $[0, 1]$.

Uniformly distributed sequences have applications in numerical integration. Obviously, all interior points of the segment $[a, b]$ are limit points for $\{x_n\}$, and moreover,

$$a = \underline{\lim_{n\to\infty}}\, x_n, \quad b = \overline{\lim_{n\to\infty}}\, x_n.$$

Given any function $f(x)$ continuous on $[a, b]$, and a sequence $\{x_n\}$'

uniformly distributed on $[a, b]$, the following relationship holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Conversely, if this equation is satisfied for all functions continuous on $[a, b]$, then the sequence $\{x_n\}$ is uniformly distributed.

7. Recurrent sequences

A sequence $\{x_n\}$ is said to be given by a *recurrence formula* if the first few terms are given and a formula is known, with the aid of which x_n is expressible in terms of the preceding terms:

$$x_n = f(x_{n-1}, x_{n-2}, \ldots, x_{n-p}), \qquad p \geqq 1 \ (n = 1, 2, \ldots).$$

For instance, $x_1 = 1$, $x_2 = 2$, $x_n = x_{n-1} = x_{n-2}\,(n = 3, 4, \ldots)$.

The sequence itself is sometimes described as *recurrent.*

The simplest example of a recurrent sequence is an *iterative sequence* $\{x_n\}$:

$$x_n = f(x_{n-1}).$$

Iterative, and in general recurrent, sequences are of great value in approximation methods – for example, in the method of successive approximations and Newton's method.

(a) *The method of successive approximations* for solving the equation $x = f(x)$, where $f(x)$ is a continuous function, leads to an iterative sequence $\{x_k\}$:

$$x_{k+1} = f\{x_k\} \qquad (k = 0, 1, 2, \ldots),$$

where some arbitrary number is taken as x_0. Here, if the sequence $\{x_k\}$ is convergent to $\bar{x}$, $\bar{x}$ proves to be a solution of the equation: $\bar{x} = f(\bar{x})$.

(b) *Newton's method* (*or method of tangents*) for finding the roots of the equation $f(x) = 0$, where $f(x)$ is a differentiable function, also leads to an iterative sequence $\{x_k\}$:

$$x_{k+1} = \frac{f'(x_k)x_k - f(x_k)}{f'(x_k)} \qquad (k = 0\ 1, 2, \ldots),$$

where some number is taken as x_0; here, if $\{x_k\}$ is convergent, then $\lim\limits_{k\to\infty} x_k = \bar{x}$ is the required root, i.e. $f(\bar{x}) \equiv 0$.

(See above, for a sufficient condition for the convergence of the sequence $\{x_n\}$ indicated in method (a).).

8. The symbols $o(\alpha_n)$ and $O(\alpha_n)$

A variable that takes a certain sequence $\{\alpha_n\}$ of values is called a *variant*. For instance, the variable term of any progression is a variant.

If α_n and β_n are given variants, while their ratio α_n/β_n tends to zero as $n \to \infty \left(\lim\limits_{n\to\infty} \alpha_n/\beta_n = 0\right)$, we say that α_n (β_n) is an *infinitesimal* (*infinitely large quantity*) with respect to β_n (α_n) and we write symbolically:

$$\alpha_n = o(\beta_n).$$

For example, $1/n^2 = o(1/n)$, $n = o(n^2)$.

If α_n, β_n are infinitesimals and $\alpha_n = o(\beta_n)$, we say that α_n (β_n) is an *infinitesimal of higher* (*lower*) *order* with respect to β_n (α_n), or α_n (β_n) *decreases faster* (*more slowly*) than β_n (α_n). If α_n, β_n are infinitely large quantities, and $\alpha_n = o(\beta_n)$, we say that α_n (β_n) *increases more slowly* (*faster*) than β_n (α_n).

If $|\alpha_n| \leqq C\,|\beta_n|$, $C > 0$ (C is a constant), we say that β_n has *a rate of decrease not faster than* α_n or that α_n has *a rate of increase not faster than* β_n, and we write symbolically:

$$\alpha_n = O(\beta_n).$$

In particular, if $\lim\limits_{n\to\infty} \alpha_n/\beta_n = C \neq 0$, then $\alpha_n = O(\beta_n)$ or $\beta_n = O(\alpha_n)$.

For example, $n = O\left(\sqrt{n^2+1}\right)$.

The equation

$$\alpha_n = O(1)$$

implies that the sequence $\{\alpha_n\}$ is bounded, i.e. that $|\alpha_n| \leqq C$ for all n.

THEOREM 3. *Given an arbitrary sequence $\{X_n\} = \{x_m^n\}$ of the following sequences:*

$$X_1 = \{x_1^1, x_2^1, x_3^1, \ldots, x_m^1, \ldots\},$$
$$X_2 = \{x_1^2, x_2^2, x_3^2, \ldots, x_m^2, \ldots\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$X_n = \{x_1^n, x_2^n, x_3^n, \ldots, x_m^n, \ldots\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

there exists a numerical sequence $X = \{x_k\} = \{x_1, x_2, \ldots, x_k, \ldots\}$, *increasing faster (decreasing faster) than any of the sequences* $\{x^n_m\}$.

For example, if $X_n = \{n^m\} = \{n^1, n^2, n^3, \ldots, n^m, \ldots\}$, the sequence $X = \{m!\} = \{1!, 2!, 3!, \ldots, m!, \ldots\}$ increases at a faster rate than any sequence $\{X_n\}$: $\lim\limits_{m \to \infty} n^n/m! = 0$ for any n.

9. Limit of a function

Let the function $f(x)$ be defined on some set X. We say that the number A is the *limit* of $f(x)$ as $x \to x_0$:

$$A = \lim_{x \to x_0} f(x),$$

or that $f(x)$ *tends to* A as $x \to x_0$, if, given any sequence $\{x_n\} \in X$, convergent to x_0 ($\lim\limits_{n \to \infty} x_n = x_0$), the sequence $\{f(x_n)\}$ is convergent to A. Or, in other words, the number A is the limit of the function $f(x)$ as $x \to x_0$:

$$A = \lim_{x \to x_0,\ x \in X} f(x) \qquad (f(x) \to A \quad \text{as} \quad x \to x_0),$$

if, given any positive number ε, there exists $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for all $x \in X$ such that $0 < |x - x_0| < \delta$. For example,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{\tan x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2},$$

$$\lim_{x \to 0} x^\alpha \ln x = 0 \quad \text{for any } \alpha > 0.$$

10. Right and left continuity of a function

We shall consider functions $y = f(x)$, defined on sets X of E_1; X will usually be a set such as

$$[a, b],\ (\alpha, \beta),\ [a, b),\ (\alpha, \beta],\ (-\infty, +\infty),\ (-\infty, 0],$$
$$[0, +\infty),\ (-\infty, 0),\ (0, +\infty),$$

etc.

Let $f(x)$ be defined on an interval (x_0, a). The number $A = f(x_0 + 0)$ is called the *limit of the function* $f(x)$ *from the right at the point* $x_0 = x_0$

if, given any sequence $\{x_n\}$ of (x_0, a), convergent to x_0, $f(x_n)$ is convergent to A. We can similarly define at the point $x = x_0$ the *limit from the left* $f(x_0-0) = B$ of $f(x)$, defined in an interval (b, x_0). For example, we have for the function $y = E(x)$ (see § 2, sec. 1) and for the integral argument $x = n$: $E(n-0) = n-1$, $E(n+0) = n$. When the argument $x_0 = 0$, the limits from the left and right of $f(x)$ are written respectively as $f(-0)$ and $f(+0)$. For example, if $f(x) = \operatorname{sign} x$ (see § 2, sec. 1), then $f(-0) = -1$, $f(+0) = +1$.

A function $f(x)$, defined at a point $x = x_0$, is *continuous there from the right (left)* if $f(x_0+0)$ $[f(x_0-0)]$ exists, equal to $f(x_0)$.

11. Continuous and discontinuous functions

If

$$f(x_0-0) = f(x_0+0) = f(x_0), \tag{1.12}$$

the function $f(x)$ is said to be *continuous* at the point $x = x_0$. If $f(x)$ is not defined in the interval (b, x_0) or (x_0, a), the left- (or right-) hand term in (1.12) is ignored. (See also p. 59). If $f(x)$, defined on the set X, is continuous at every point $x \in X$, it is said to be *continuous* on the set X.

For instance, $f(x) = \sqrt{(x)}$ is continuous for all $x \geqq 0$.

Otherwise, $f(x)$ is described as *discontinuous*. We say that ($f(x)$ has a *discontinuity of the first kind* or *jump* at the point $x = x_0$ if $f(x_0-0)$ and $f(x_0+0)$ exist, but (1.12) is not satisfied. In all other cases of a discontinuous function, the point $x = x_0$ is called a *point of discontinuity of the second kind.* For instance, the function

$$f(x) = \begin{cases} -1 & \text{for} \quad |x| < 1, \\ +1 & \text{for} \quad |x| \geqq 1 \end{cases}$$

has a discontinuity of the first kind at the points $x_1 = -1$ and $x_2 = +1$.

Certain commonly encountered functions $y = f(x)$ are equal, at a point of discontinuity of the first kind $x = x_0$, to the arithmetic mean value

$$f(x_0) = \frac{f(x_0-0)+f(x_0+0)}{2}.$$

For example, in the case of $f(x) = \operatorname{sign} x$ (see § 2, sec. 1):

$$f(0) = \frac{f(-0)+f(+0)}{2}.$$

We say that $f(x)$ is *uniformly continuous* on the set X if, given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for *any* pair of points $x', x'' \in X$ for which $f(x)$ has a meaning, $|x'-x''| < \delta$ implies $|f(x')-$ $-f(x'')| < \varepsilon$. A function $f(x)$, defined and continuous on a bounded segment $[a, b]$, is uniformly continuous on this segment (Cantor's theorem).

12. Functional sequences

An important role is played in analysis by *functional sequences* $\{f_n(x)\}$ $(n = 1, 2, \ldots)$, defined on some set X of the numerical axis E_1. Various definitions can be given of passage to the limit for such sequences. It is natural to start from passage to the limit at each point, when $\{f_n(x)\}$ becomes a numerical sequence for any fixed $x \in X$. If the sequence $\{f_n(x)\}$ is convergent as $n \to \infty$ for any $x \in X$, its limit depends on the point $x \in X$, i.e. is a function $f(x)$ (a *limit function*), and this is written as

$$f_n(x) \xrightarrow[n\to\infty]{} f(x) \quad \text{or} \quad \lim_{n\to\infty} f_n(x) = f(x) \quad x \in X.$$

EXAMPLE 21.

$$\lim_{n\to+\infty} \arctan nx = \frac{\pi}{2}\operatorname{sign} x = \begin{cases} -\frac{\pi}{2} & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x = 0, \\ \frac{\pi}{2} & \text{for} \quad x > 0. \end{cases}$$

This example shows that the limit of continuous functions may be a discontinuous function.

As a result of a double passage to the limit, functions can be obtained in the limit that have even more complicated discontinuities, for instance,

$$\lim_{m\to\infty} \lim_{n\to\infty} (\cos 2\pi m!\, x)^n = \chi(x),$$

where $\chi(x)$ is *Dirichlet's function:*

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational.} \end{cases}$$

REMARK. Together with functional sequences, we often encounter in analysis sequences of numbers, dependent on functions *(functionals)* For instance, the mean values of functions are defined by the limits of such sequences.

If the function $f(x)$ is integrable on $[a, b]$ and $f_{\nu n} = f(a+\nu\delta_n)$, $\delta_n = \dfrac{b-a}{n}$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_{kn} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

is the *arithmetic mean* of $f(x)$ on $[a, b]$;

$$\lim_{n\to\infty} \sqrt[n]{f_{1n}f_{2n}\cdots f_{nn}} = \exp\left\{ \frac{1}{b-a} \int_a^b f(x)\,dx \right\}$$

is the *geometric mean* of $f(x)$ on $[a, b]$, and

$$\lim_{n\to\infty} n \left(\sum_{k=1}^{n} \frac{1}{f_{kn}} \right)^{-1} = (b-a) \left(\int_a^b \frac{dx}{f(x)} \right)^{-1}$$

is the *harmonic mean* of $f(x)$ on $[a, b]$.

The following also holds:

THEOREM 4. *If functions $f(x)$ and $g(x)$ are continuous and positive on* $[a, b]$, then

$$(1) \qquad \lim_{n\to\infty} \sqrt[n]{\int_a^b g(x)\,[f(x)]^n\,dx} = \max_{x\in[a,b]} f(x),$$

$$(2) \qquad \lim_{n\to\infty} \frac{\int_a^b g(x)\,[f(x)]^{n+1}\,dx}{\int_a^b g(x)\,[f(x)]^n\,dx} = \max_{x\in[a,b]} f(x).$$

13. Uniform convergence of functions

The concept of a *uniformly convergent sequence of functions* $\{f_n(x)\}$ plays an exceptionally important role in analysis.

DEFINITION. *A sequence of functions $\{f_n(x)\}$, defined on a set $X \subset E_1$, is said to converge uniformly as $n \to \infty$ to the function $f(x)$, also defined on the set X, if, given any positive number ε, an integer*

$N = N(\varepsilon)$ can be found, independent of $x \in X$, such that, for all $n > N$:

$$|f_n(x) - f(x)| < \varepsilon. \tag{1.13}$$

If $\{f_n(x)\}$ is convergent as $n \to \infty$ at every point $x \in X$, but does not satisfy the uniformity condition (1.13), we say that $\{f_n(x)\}$ converges *non-uniformly* to $f(x)$ on the set X. With non-uniform convergence, the number N depends not only on the choice of ε, but also on the number $x \in X$:

$$N = N(\varepsilon, x).$$

THEOREM 5. *If all the functions $f_n(x)$ of a sequence $\{f_n(x)\}$, uniformly convergent to $f(x)$ on X as $n \to \infty$ are continuous, the limit function $f(x)$ is also continuous on X.*

It follows from this that, if the limit function is discontinuous, the convergence as $n \to \infty$ of the sequence $\{f_n(x)\}$ is non-uniform.

EXAMPLE 22. The sequence $\{f_n(x)\} = \{x^n\}$ is uniformly convergent as $n \to \infty$ to $f(x) = 0$ on any segment $[0, q]$, $0 < q < 1$; on the segment $[0, 1]$ this sequence converges non-uniformly to the function

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \leqq x < 1, \\ 1 & \text{for} \quad x = 1. \end{cases}$$

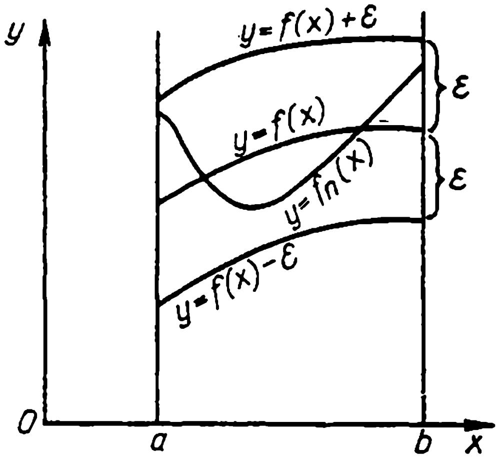

FIG. 1.

CAUCHY'S TEST. A *necessary and sufficient condition for a sequence of functions $f_n(x)$, defined on a set $X \subset E_1$, to be uniformly convergent as $n \to \infty$ to a function $f(x)$ is that, given any $\varepsilon > 0$, there exists an N, depending only on ε, such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $x \in X$, provided only that $n > N$ and $m > N$.*

Geometrical interpretation of uniform convergence. Let $f_n(x)$ ($n = 1, 2, \ldots$) be continuous in $[a, b]$, and let the sequence $\{f_n(x)\}$ be uniformly convergent as $n \to \infty$ to the function $f(x)$, continuous in $[a, b]$. Now, all the curves $y = f_n(x)$ fall within an ε-neighbourhood of the curve $y = f(x)$ when $n > N$ (see Fig. 1), i.e. they are contained in the strip between the curves $y = f(x) - \varepsilon$ and $y = f(x) + \varepsilon$.

14. Convergence in the mean

A functional sequence $\{f_n(x)\}$ is *convergent in the mean* on the segment $[a, b] \subset E_1$ as $n \to \infty$ to the function $f(x)$ if, given any $\varepsilon > 0$, there is a number N such that, for all $n > N$, we have

$$\int_a^b [f_n(x) - f(x)]^2 \, dx < \varepsilon$$

(it is assumed that this integral exists).

Use is often made of convergence in the mean in various branches of analysis (for instance, in the approximation methods of analysis).

Convergence, defined by the norm of a space (see Chapter II, §1, sec. 2), is a generalization of convergence in the mean.

15. The symbols $o(x)$ and $O(x)$

Given two functions $x(t)$ and $y(t)$, defined on a set X, if their ratio $x(t)/y(t)$ tends to zero as $t \to a \in X$ ($\lim\limits_{t \to a} x(t)/y(t) = 0$), we say that $x(t)$ $[y(t)]$ is an *infinitesimal* (*infinitely large quantity*) *with respect to* $y(t)$ $[x(t)]$ and we write symbolically:

$$x(t) = o(y(t)).$$

For example, $t^2 = o(\sin t)$ as $t \to 0$, $t^n = o(e^t)$ as $t \to \infty$ for any $n > 0$.

If $x(t)$ and $y(t)$ are infinitesimals as $t \to a$, and $x(t) = o(y(t))$, we say that $x(t)$ ($y(t)$) is an *infinitesimal of higher* (*lower*) *order with respect to* $y(t)$ ($x(t)$), or: $x(t)$ ($y(t)$) *decreases faster* (*more slowly*) than $y(t)$ ($x(t)$).

If $x(t)$, $y(t)$ are infinitely large quantities as $t \to a$, and $x(t) = o(y(t))$, we say that $x(t)$ ($y(t)$) *increases more slowly* (*faster*) than $y(t)$ ($x(t)$).

If $|x(t)| \leqq C|y(t)|$ (C is a positive constant), we say that $x(t)$ *does not decrease at a faster rate* than $y(t)$ or that $x(t)$ *does not increase at a faster rate* than $y(t)$, and we write symbolically:

$$x(t) = O(y(t)).$$

For instance, $t = O(t\sin(1/t))$ and $t = O(\tan 2t)$ as $t \to 0$, $e^t = O\left(e^{\frac{1}{2}t}\right)$ and $e^t = O\left(e^{\sqrt{t^2+1}}\right)$ as $t \to \infty$.

In particular, if $\lim\limits_{t \to a} x(t)/y(t) = C \neq 0$ (C is a constant), then

$$x(t) = O(y(t)) \quad \text{and} \quad y(t) = O(x(t)).$$

THEOREM 6. *Whatever the sequence of functions $f_n(x)$, defined in the neighbourhood of a point $x = a$, there always exists a function $\varphi(x) \bar{\in} M$, decreasing (increasing) faster then any of the $f_n(x)$ as $x \to a$.*

16. Monotonic functions

A function $f(x)$ is described as *monotonically non-increasing (non-decreasing)* on a set X (e.g. on $[a, b]$), if, given any points $x_1, x_2 \in X$ such that $x_1 < x_2$, we have $f(x_1) \geqq f(x_2)$ ($f(x_1) \leqq f(x_2)$). A function $f(x)$ is said to be *monotonically increasing (decreasing) in the strict sense* (or to be *strictly monotonic*) if $f(x_1) < f(x_2)$ ($f(x_1) > f(x_2)$) for any $x_1, x_2 \in X$ for which $x_1 < x_2$.

Functions monotonically non-increasing, monotonically non-decreasing, and strictly monotonic are all described as *monotonic*.

When $f(x)$ is monotonic, it always has limits from the left and right at a point of discontinuity $x = x_0$; if $f(x)$ is non-increasing, we have

$$f(x_0-0) \geqq f(x_0) \geqq f(x_0+0);$$

if $f(x)$ is non-decreasing,

$$f(x_0-0) \leqq f(x_0) \leqq f(x_0+0).$$

Let $y = f(x)$ be a monotonically increasing (decreasing) continuous function, defined in a segment interval or semi-interval $X \in E_1$, and mapping X into a segment, interval or semi-interval $Y \subset E_1$; an inverse $x = \varphi(y)$ of $f(x)$ now exists, defined on Y. The function

$\varphi(y)$ is continuous on Y and is monotonically increasing or decreasing along with $f(x)$. For example, the function $y = x^2$ in the semi-interval $(0, +\infty)$ has an inverse $x = \sqrt{y}$ in $(0, +\infty)$.

17. Convex functions

A function $f(x)$, defined on a set X (a finite or infinite interval, semi-interval, segment) is described as *convex* if, given any numbers $x_1, x_2 \in X$, we have

$$f\left(\frac{x_1+x_2}{2}\right) \leqq \frac{1}{2}[f(x_1)-f(x_2)];$$

$f(x)$ is described as *concave* if the reverse inequality is satisfied:

$$f\left(\frac{x_1-x_2}{2}\right) \geqq \frac{1}{2}[f(x_1)-f(x_2)].$$

This means geometrically that, if $f(x)$ is convex, no arc of the curve $y = f(x)$ lies above the chord subtending it, whereas if $f(x)$ is concave, no arc of the curve $y = f(x)$ lies below the chord subtending it.

For example, the functions $|x|$, x^2, e^x, $x+|x|$ are convex, while the functions $-|x|$, $\sqrt{x}$, $-e^{-x}$ are concave; the function $y = x^\alpha \ (x > 0)$ is convex $\alpha \geqq 1$ and $\alpha < 0$ and concave for $0 < \alpha < 1$. A necessary and sufficient condition for a function $f(x)$ to be convex on $[a, b]$ is that, given any numbers x_1, x_2, x_3 satisfying $a \leqq x_1 < x_2 < x_3 \leqq b$, the following inequality holds:

$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} \geqq 0.$$

If $f(x)$ is continuous (even only from the right) and $f(x) = \frac{1}{2}[f(x-h)+f(x+h)]$, it is convex.

If $f(x)$ is monotonic in $[a, b]$, $\int_a^x f(\xi)\, d\xi$ is a convex function.

It must be borne in mind that the graph of a convex function faces downwards, i.e. it is a concave curve in the accepted geometrical terminology.

Convex functions have the following properties:

1°. A convex (concave) function $f(x)$ is continuous at every point of its domain of definition.

2°. If $f(x)$ is convex, $-f(x)$ is concave on the set X.

3°. A convex (concave) function $f(x)$, not equal to a constant on the segment $[a, b]$, cannot attain a maximum (minium) inside $[a, b]$.

4°. If $f(x)$ is convex (concave) on $[a, b]$, and $l(x)$ is a linear function, where $l(a) = f(a)$ and $l(b) = f(b)$, then either $f(x) < l(x)$ $(f(x) < l(x))$ at every point of (a, b), or $f(x) \equiv l(x)$.

5°. A linear combination of convex functions with positive coefficients is a convex function; in particular, the sum of a finite number of convex functions is a convex function.

6°. If $f(u)$ is a non-decreasing convex function, while $u = \varphi(x$ is a convex function, $f[\varphi(x)]$ is a convex function of x.

7°. The inverse of a decreasing (increasing) convex function is a convex (concave) function.

8°. If $f(x)$ is convex on $[a, b]$, given any positive numbers $p_i > 0$ $(i = 1, 2, \ldots, n)$ such that $p_1+p_2+\ldots+p_n = 1$ and any points x_1, $x_2, \ldots, x_n$ on the segment $[a, b]$, we have

$$f(p_1x_1+p_2x_2+ \ldots +p_nx_n) \leqq p_1f(x_1)+p_2f(x_2)+ \ldots +p_nf(x_n).$$

9°. Any convex function $f(x)$, satisfying the condition $f(x_0) = 0$, is expressible in the form

$$f(x) = \int_{x_0}^{x} p(t)\,dt,$$

where $p(t)$ is a non-decreasing function, continuous from the right.

A function $\varphi(x)$, defined on a set $X \subset E_1$, is described as *logarithmically convex* if $\log \varphi(x)$ is a convex function, i.e. if

$$\varphi^2\left(\frac{x_1-x_2}{2}\right) \leqq \varphi(x_1)\varphi(x_2)$$

for any $x_1, x_2 \in X$. A logarithmically convex function is a convex function. For instance, the function $\varphi(x) = x \log x$ is logarithmically convex for $x > 0$; $\varphi(x) = e^{x^2}$ is logarithmically convex for all $x \in$

$\in(-\infty, +\infty)$; $\varphi(x) = x^k$ (k is an integer) is logarithmically convex for $x \in (-\infty, +\infty)$ and even $k \neq 0$, for $x \in (0, +\infty)$ and any $k \neq 0$; $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ is logarithmically convex for $x > 0$ (concerning the function $\Gamma(x)$, see Chapter VI, § 4, sec. 5).

CHAPTER II

n-DIMENSIONAL SPACES AND FUNCTIONS DEFINED THERE

Introduction

We considered in Chapter I functions of a single variable as functions of a point of the numerical axis E_1. Similarly, functions $f(x, y)$ of two variables can be regarded as functions of a point of the plane E_2 with coordinates x, y, and functions $f(x, y, z)$ of three variables as functions of a point of space E_3 with coordinates x, y, z.

About the middle of the last century, coordinate space of n dimensions was introduced into mathematics, and functions of n variables came to be regarded as functions of a point of n-dimensional space. At the same time, various concepts of ordinary two- and three-dimensional geometry were generalized to n-dimensional spaces.

This generalization proved to be not merely formal. Our geometrical intuition can be carried over to multi-dimensional entities, and the treatment in geometrical terms of the problems of analysis and algebra of n dimensions led to greater clarity. Geometrical intuition sometimes enables us to discover the facts in n-dimensional geometry, which can be interpreted as corresponding facts of analysis and algebra.

In the present chapter § 1 deals with the theory of n-dimensional spaces, and in particular with the theory of orthogonal systems of vectors, which is an elementary analogue of the more complicated theory of systems of orthogonal functions, discussed in Chapter IV.

Section 2 is devoted to passage to the limit, continuous functions and continuous (in the generalized sense) operators on them in n-dimensional space. It is directly related to Chapter I, in which we discussed the same topics for the particular case $n = 1$.

Section 3 gives a treatment of one of the branches of n-dimensional geometry — the theory of convex n-dimensional bodies, which, apart from its geometrical interest, has acquired importance in a number of applied mathematical problems. It should be remarked that no great simplification is obtained by confining the discussion of convex solids to three-dimensional rather than n-dimensional space; and in fact, it is precisely the theory of convex bodies in space of a large number of dimensions that has acquired practical applications.

§ 1. n-dimensional spaces

1. n-dimensional coordinate space

An *element X of an n-dimensional space E_n* is defined as a set of n numbers $x_1, x_2, \ldots, x_n$. It is written as $X(x_1, x_2, \ldots, x_n)$ or $X = (x_1, x_2, \ldots, x_n)$. The space E_n is the set of such elements.

Elements of the space E_n are treated in two ways: on the one hand, they are regarded as *points with coordinates* $x_1, x_2, \ldots, x_n$, on the other, as *vectors with coordinates* $x_1, x_2, \ldots, x_n$ (*n-dimensional vector space*). We shall start by using the first treatment.

There are various ways of introducing *distance* or a *metric* into the space E_n (see, for example below, § 3, sec. 3). The most common is the following (Euclidean) metric: *the distance $\varrho(X, Y)$ between the points $X(x_1, x_2, \ldots, x_n)$ and $Y(y_1, y_2 \ldots, y_n)$ of E_n* is defined as the number

$$\varrho(X, Y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\,. \tag{2.1}$$

The space E_n with a distance introduced in this way is called *n-dimensional Euclidean space.*

Formula (2.1) reduces to the formula for the distance between two points in analytical geometry when $n = 1, 2, 3$.

We write θ for the point with zero coordinates: $\theta = (0, 0, \ldots, 0)$ (the origin of coordinates). We have

$$\varrho(X, \theta) = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{2.2}$$

The distance thus introduced has the following properties:

(1) $\varrho(X, Y) \geqq 0$, where $\varrho(X, Y) = 0$ only when $X = Y$;
(2) $\varrho(X, Y) = \varrho(X, Y)$;
(3) $\varrho(X, Y) \leqq \varrho(X, Y) + \varrho(Y, Z)$

(the *triangle inequality*).

An attempt is generally made to preserve these properties in generalized concepts of distance.

2. *n*-dimensional vector space

The n-dimensional space E_n can also be regarded as a *vector space.* The operations of addition and multiplication by a scalar (number) can be performed on vectors in the plane E_2 and in three-dimensional space E_3. Every element $X(x_1, x_2, \ldots, x_n)$ of E_n will be regarded as *a vector with coordinates* $x_1, x_2, \ldots, x_n$.

The *sum of two vectors* $X(x_1, x_2, \ldots, x_n)$ and $Y(y_1, y_2, \ldots, y_n)$ is the vector $X+Y$ of E_n with coordinates $x_1+y_1, x_2+y_2, \ldots, x_n+y_n$. The *difference* $X-Y$ is similarly defined.

If λ is a number (scalar), and $X(x_1, x_2, \ldots, x_n)$ is a vector of E_n, λX denotes the vector of E_n with coordinates $\lambda x_1, \lambda x_2, \ldots, \lambda x_n$. Vectors X and λX are described as *collinear. Linear operations on vectors reduce to linear operations on their coordinates.*

The zero point θ corresponds to the zero vector θ with components equal to zero. Obviously, $X+\theta = X$, $\lambda\theta = \theta$.

The vectors

$$e_1(1, 0, 0, \ldots, 0),\ e_2(0, 1, 0, \ldots, 0),\ \ldots,\ e_n(0, 0, \ldots, 0, 1)$$

or alternatively,

$$e_i(\delta_{i1}, \delta_{i2}, \ldots, \delta_{ij}, \ldots, \delta_{in}),$$

where δ_{ij} is the *Kronecker delta:*

$$\delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j, \\ 1 & \text{for} \quad i = j, \end{cases}$$

are called *unit vectors.* The following equation holds for any vector $X = (x_1, x_2, \ldots, x_n)$:

$$X = \sum_{i=1}^{n} x_i e_i.$$

The *norm* $||X||$ of the vector X is the number

$$||X|| = \sqrt{\sum_{i=1}^{n} x_i^2}, \tag{2.3}$$

equal to the distance $\varrho(X, \theta)$ from the point X to θ.

The norm of a vector satisfies the conditions:

$$\left.\begin{array}{ll} (1) & ||X|| \geqq 0, \quad \text{where} \quad ||X|| = 0 \text{ only when } X = \theta; \\ (2) & ||\lambda X|| = |\lambda|\,||X||; \\ (3) & ||X+Y|| \leqq ||X|| + ||Y||. \end{array}\right\} \tag{2.4}$$

The distance $\varrho(X, Y)$ between elements X and Y, introduced in accordance with (2.1), is the same as the norm of the difference between the corresponding vectors

$$\varrho(X, Y) = ||X-Y||. \tag{2.5}$$

The *sphere* $S(X_0, r)$ in E_n of radius r with centre at X_0 is the set of points (vectors) X, for which

$$\varrho(X, X_0) \equiv ||X-X_0|| < r.$$

If we introduce for sets in E_n the notation introduced in Chapter I, § 2, sec. 2, we have

$$S(X_0, r) = (\varrho(X, X_0) < r).$$

3. Scalar product

In analogy with the two- and three-dimensional case, the *scalar* or *inner product of two vectors* $X(x_1, x_2, \ldots, x_n)$ and $Y(y_1, y_2, \ldots, y_n)$ of E_n is defined by the equation

$$XY = (XY) = \sum_{i=1}^{n} x_i y_i. \tag{2.6}$$

The elementary properties of scalar products are:

(1) $XY = YX;$

(2) $XX = ||X||^2 \geqq 0.$

CAUCHY'S INEQUALITY. *Given any two vectors* $X(x_1, x_2, \ldots, x_n)$ *and* $Y(y_1, y_2, \ldots, y_n)$ *of* E_n, *we have*

$$|XY| \leqq ||X||\,||Y|| \tag{2.7}$$

or

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \sqrt{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2}. \tag{2.7'}$$

Equality holds in formulae (2.7) *and* (2.7′) *if and only if the vectors* X *and* Y *are collinear* (i.e. if $X = \theta$ or if $Y = \lambda X$, i.e. if all the $x_i = 0$ or $y_i = \lambda x_i$ $(i = 1, 2, \ldots, n)$).

The angle between two vectors. Let $X(x_1, x_2, \ldots, x_n)$ and $Y(y_1, y_2, \ldots, y_n)$ be two vectors of E_n different from θ; the *angle* φ between them is given by

$$\cos\varphi = \frac{xy}{\|x\|\,\|y\|}. \tag{2.8}$$

In view of (2.7), $|\cos\varphi| \leqq 1$.

The projection of the vector X on to the vector Y is given in magnitude by

$$\frac{xy}{\|y\|} = \|x\| \cos\varphi. \tag{2.9}$$

When $\|Y\| = 1$, the projection of vector X on to Y is equal in magnitude to their scalar (inner) product:

$$XY = \|X\| \cos\varphi. \tag{2.10}$$

EXAMPLE 1. The magnitude of the projection of the vector $X(x_1, x_2, \ldots, x_n)$ on to the unit vector e_i $(i = 1, 2, \ldots, n)$ is equal to the coordinate x_i of the vector X.

4. A linear system and its basis

An n-dimensional vector space is a particular case of an n-dimensional linear system.

A *linear system L* is a set of elements to which we can apply the operation of addition, i.e. the operation of finding a new element $c \in L$ from two elements $a, b \in L$, $c = a+b$, and the operation of multiplication by a real number λ, i.e. the operation of finding from an element $a \in L$ and a number λ an element $d = \lambda a \in L$. These operations have the following properties:

(1) $(a+b)+c = a+(b+c)$;
(2) $a+b = b+a$;
(3) $\lambda(\lambda_1 a) = (\lambda\lambda_1)a$;
(4) $\lambda(a+b) = \lambda a+\lambda b$;
(5) $(\lambda+\lambda_1)a = \lambda a+\lambda_1 a$;
(6) $1.a = a$.

The system L possesses a zero element θ, for which $a+\theta = a, 0.a = \theta$ for $a \in L$.

Linearly independent elements. Elements $x_1, x_2, \ldots, x_k$ of a linear system are said to be *linearly dependent* if there exists a set of numbers $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1x_1+c_2x_2+ \ldots +c_kx_k = \theta; \tag{2.11}$$

if there is no such set of numbers, i.e. if (2.11) is satisfied only with

$$c_1 = c_2 = \ldots = c_k = 0,$$

the elements $x_1, x_2, \ldots, x_k$ are said to be *linearly independent*.

EXAMPLE 2. On a plane, the vectors X_1 (1, 1) and X_2 (2, 3) are linearly independent, while the vectors $X_1(1, 1)$ and X_3 (3, 3) are linearly dependent (since $3X_1-X_3 = \theta$). In three-dimensional space, the vectors Y_1 (1, 0, 0), Y_2 (1, 1, 0) and Y_3 (1, 1, 1) are linearly independent; the vectors Z_1 (1, 0, 0), Z_2 (2, 1, 1) and Z_3 (3, 2, 2) are linearly dependent since

$$Z_1-2Z_2+Z_3 = 0$$

A linear system is described as *n-dimensional* if it contains n linearly independent vectors, and any $n+1$ elements of it are linearly dependent. A linear system is said to be *infinite-dimensional* if it contains any number of linearly independent elements. An n-dimensional vector space is an n-dimensional linear system.

The set of solutions of a linear homogeneous differential equation forms a linear system. In the case of an ordinary nth-order equation this system is n-dimensional; in the case of a partial differential equation the system is infinite-dimensional.

The *basis* of an n-dimensional linear system L_n is any set $(e_1, e_2, \ldots, e_n)$ of n linearly independent elements.

Any element l of the system L_n is linearly (and uniquely) expressible in terms of the elements of the basis:

$$l = x_1e_1 + x_2e_2 + \ldots + x_ne_n; \tag{2.12}$$

the elements x_ie_i ($i = 1, 2, \ldots, n$) are called the *components with respect to the basis* ($e_1, e_2, \ldots, e_n$) while the numbers x_n are the *coordinates with respect to this basis.*

EXAMPLE 3. The unit vectors e_i form a basis in n-dimensional vector space:

$$e_1 = (1, 0, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots$$
$$\ldots, e_n = (0, 0, \ldots, 0, 1).$$

An n-dimensional linear system can be regarded as an n-dimensional space, in which the elements of the basis play the part of unit vectors.

Examples of n-dimensional linear systems. Linear systems of functions, for which addition and multiplication by a number are understood in the ordinary sense, are aften encountered in analysis. We shall quote some examples.

(1) The $(n+1)$-dimensional system of polynomials

$$P(x) = \sum_{k=0}^{n} c_kx^k$$

of degree not higher than n. The system of powers

$$1, x, x^2, \ldots, x^n,$$

can be taken as the basis, as also can any system of polynomials

$$P_0 = 1, \quad P_1(x) = x + a_{10},$$

$$P_2(x) = x^2 + a_{21}x + a_{20}, \ldots, P_k(x) = x^k + \sum_{i=0}^{k-1} a_{ki}x^i, \ldots,$$
$$\ldots, P_n(x) = x^n + \sum_{i=0}^{n-1} a_{ni}x^i.$$

(2) The system of homogeneous polynomials $P_k(x_1, x_2, \ldots, x_n)$ of degree k in n variables $x_1, x_2, \ldots, x_n$, i.e. sums of terms of the form

$$a_{k_1k_2\ldots k_n}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n},$$

where

$$k_1 + k_2 + \ldots + k_n = k.$$

The basis of this system consists of all single terms of the form $x_1^{k_1}x_2^{k_2}\dots x_n^{k_n}$ of degree $k = k_1+k_2+\dots+k_n$. The number of such single terms (i.e. the number of dimensions of the system) is

$$\frac{(n+k-1)!}{n!\,(k-1)!}.$$

For instance, with $n = 3$, $k = 3$, the number of them is equal to $5!/3!2! = 10$ (the single terms are x_1^3, $x_1^2x_2$, $x_1^2x_3$, $x_1x_2^2$, $x_1x_2x_3$, $x_1x_3^2$, x_2^3, $x_2^2x_3$, $x_2x_3^2$, x_3^3).

If a system L_n has one basis $(e_1, e_2, \dots, e_n)$, it has an infinite set of bases. Let $(e_1', e_2', \dots, e_n')$ be another basis; then the elements e_i of the old basis are expressible in terms of the elements e_j' of the new in accordance with

$$e_i = \sum_{j=1}^{n} \alpha_{ij}e_j' \qquad (i = 1, 2, \dots, n),$$

where the determinant $|\alpha_{ij}|_{i,j}^{1,n} \neq 0$. An element l is expressible in terms of the elements of the new basis in accordance with

$$l = \sum_{i=1}^{n} x_i'e_i'.$$

The connection between the coordinates x_i and x_i' of an element l with respect to bases (e_i) and (e_i') is given by

$$x_i = \sum_{j=1}^{n} \alpha_{ij}x_j' \qquad (i = 1, 2, \dots, n). \tag{2.13}$$

It follows from what has been said that the elements of any n-dimensional linear system L_n with basis $(e_1, e_2, \dots, e_n)$ can be regarded as vectors of an n-dimensional space in which the elements e_i of the basis play the role of unit vectors. We shall confine our attention to such spaces in the present chapter.

5. Linear functions

A *linear function* or *linear form* in E_n is a function $f(X) = f(x_1, x_2, \dots, x_n)$, satisfying the conditions:

(1) $f(X+Y) = f(X)-f(Y)$. (2.14)

(2) $f(\lambda X) = \lambda f(X)$, where λ is any number. (2.15)

If $f(e_i) = y_i$ for the unit vectors e_i $(i = 1, 2, \ldots, n)$, then

$$f(X) = f(x_1, x_1, \ldots, x_n) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} y_i x_i. \tag{2.16}$$

The numbers y_i are the *coefficients of the linear form f.*

These coefficients can be regarded as the coordinates of the vector $Y = (y_1, y_2, \ldots, y_n)$ of E_n; we now have from (2.16):

$$f(X) = YX. \tag{2.17}$$

The linear function $f(X)$ is equal to the scalar product of the fixed vextor Y with the variable vector X. We usually say that the function $f(X)$ is *generated by the vector Y.*

The scalar product XY is a *bilinear* function of X and Y: it is a linear function of X if Y is fixed, and a linear function of Y if X is fixed.

We have the equation

$$||Y|| = \max YX. \tag{2.18}$$

i.e. *the norm of the vector Y generating the linear form YX is equal to its maximum value on the unit sphere $||X|| \leqq 1$.*

Criteria for linear independence. The Gram determinant for vectors. Let

$$X_i(x_{i1}, x_{i2}, \ldots, x_{in}) \qquad (i = 1, 2, \ldots, n)$$

be vectors of the space E_n. From them we form the following determinant:

$$\Delta(X_1, X_2, \ldots, X_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}. \tag{2.19}$$

THEOREM 1. *A necessary and sufficient condition for vectors X_1, $X_2, \ldots, X_n$ of E_n to be linearly independent is that*

$$\Delta(X_1, X_2, \ldots, X_n) \neq 0.$$

Gram's determinant $\Gamma_{X_1, X_2, \ldots, X_m}$ of the vectors $X_1, X_2 \ldots, X_m$ of E_n is given by

$$\Gamma_{X_1X_2\ldots X_m} = \begin{vmatrix} (X_1X_1) & (X_1X_2) & \ldots & (X_1X_m) \\ (X_2X_1) & (X_2X_2) & \ldots & (X_2X_m) \\ \ldots & \ldots & \ldots & \ldots \\ (X_mX_1) & (X_mX_2) & \ldots & (X_mX_m) \end{vmatrix}. \qquad (2.20)$$

Gram's determinant has the properties:

(1) $\Gamma_{X_1X_2\ldots X_m} \geqq 0$;

(2) A necessary and sufficient condition for linear independence of the vectors $X_1, X_2, \ldots, X_m$ is that

$$\Gamma_{X_1X_2\ldots X_m} > 0; \qquad (2.21)$$

(3) When $m = n$,

$$\Gamma_{X_1X_2\ldots X_n} = [\Delta(X_1, X_2, \ldots, X_n)]^2.$$

See Chapter IV, § 2, sec. 3 for the Gram determinant for functions.

Manifolds. Let $X_1, X_2, \ldots, X_k$ be linearly independent elements of E_n $(1 \leqq k < n)$.

A set of elements X of the form

$$X = \sum_{i=1}^{k} c_i x_i \qquad (2.22)$$

with arbitrary real c_i is called a *k-dimensional linear manifold.*

A one-dimensional linear manifold, i.e. a set of elements of the form $X = \lambda x_1$ $(x_1 \neq \theta)$ is a *straight line passing through θ and x_1*. Part of this straight line – the set of elements of the form $X = \lambda x_1$, $\lambda > 0$, – is called a *ray.*

A displaced k-dimensional manifold or *k-dimensional plane* is a set of elements X of the form

$$X = X_0 + \sum_{i=1}^{k} c_i X_i \qquad (2.23)$$

with fixed and linearly independent $X_1, X_2, \ldots, X_k$ and arbitrary values of the numbers $c_1, c_2, \ldots, c_k$. It is obtained by *displacement of the manifold* (2.22) *along the vector* X_0. A one-dimensional displaced manifold is a *straight line:*

$$X = X_0 + tX_1 \qquad (-\infty < t < +\infty). \qquad (2.24)$$

We have for a straight line through the points X_1 and X_2:

$$X = (1-t)X_1 + tX_2 \qquad (-\infty < t < +\infty). \qquad (2.25)$$

The *segment of the straight line joining points* X_1 *and* X_2 is the set of elements of the form

$$X = (1-t)X_1 + tX_2 \qquad (0 \leqq t \leqq 1). \tag{2.26}$$

An $(n-1)$-dimensional displaced manifold is called a *hyperplane*.

Any hyperplane is given by an equation (i.e. is a set of points $X(x_1, x_2, \ldots, x_n)$, satisfying the equation)

$$YX = \sum_{j=1}^{n} y_j x_j = d, \tag{2.27}$$

where YX is a linear form generated by the non-zero vector $Y(y_1, y_2, \ldots, y_n)$. Conversely, *every such equation defines a hyperplane*.

In general, *any k-dimensional linearly displaced manifold* $(1 \leqq k < n)$ *is given by a system of equations*

$$Y_i X = \sum_{j=1}^{n} y_{ij} x_j = d_i \qquad (i = 1, 2, \ldots, n-k), \tag{2.28}$$

where the vectors $Y_i(y_{i1}, y_{i2}, \ldots, y_{in})$ $(i = 1, 2, \ldots, n-k)$, generating corresponding linear forms, are *linearly independent*; in other words, every such manifold *is given by* $n-k$ *linear equations*. Conversely, $n-k$ *equations* (2.28) (*with linear independence of the vectors* Y_i) *define a displaced k-dimensional manifold*. If all the right-hand sides $d_i = 0$, we get a k-dimensional (non-displaced) manifold. In particular, the *straight line* (2.24) is given by $n-1$ equations (2.28).

6. Linear envelope

The *linear envelope of a set* M *of* E_n is the least linear manifold containing M: in other words, the linear envelope of the set M is the set of all linear combinations of any finite number of elements of M, i.e. of elements of the form

$$\sum_{i=1}^{r} t_i X_i, \qquad X_i \in M.$$

In particular, if $X_1, X_2, \ldots, X_k$ are linearly independent elements of E_n, their linear envelope is the k-dimensional linear manifold consisting of all elements of the form

$$\sum_{i=1}^{k} l_i X_i.$$

Notice that, if the vectors $X_1, X_2, \ldots, X_k$ form a basis of the k-dimensional linear manifold E_k of E_n, we can supplement them by vectors $Z_1, Z_2, \ldots, Z_{n-k}$ in such a way that *the vectors* $X_1, X_2 \ldots, X_k$, $Z_1, Z_2, \ldots, Z_{n-k}$ *form a basis in* E_n. Let E_{n-k} denote the linear envelope of vectors L_i. Every element Y of E_n can be expressed (uniquely) as

$$Y = X + Z, \tag{2.29}$$

where

$$X = \sum_{i=1}^{k} c_i X_i \in E_k, \quad Z = \sum_{j=1}^{n-k} d_j Z_j \in E_{n-k}.$$

Here, the space E_n is the *direct sum* of the manifolds E_k and E_{n-k}, this being written symbolically as

$$E_n = E_k + E_{n-k}. \tag{2.30}$$

X and Z in (2.29) are called the *components* of the vector Y in E_k and E_{n-k}.

7. Orthogonal systems of vectors

Two (non-zero) vectors X and Y are said to be *orthogonal* if $XY = 0$ (i.e. if $\cos\varphi = 0$). The vectors $X_1, X_2, \ldots, X_m$ form an *orthogonal system* if they are orthogonal in pairs, i.e.

$$X_i X_j = 0 \quad \text{for} \quad i \neq j. \tag{2.31}$$

The vectors of an orthogonal system are linearly independent. If, in addition, $||X_i|| = 1$, the vectors are said to form an *orthonormal system.* In this case,

$$X_i X_j = \delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j, \\ 1 & \text{for} \quad i = j. \end{cases} \tag{2.32}$$

The unit vectors $e_1, e_2, \ldots, e_n$ form an orthonormal system.

In n-dimensional space E_n there exists an infinite set of orthogonal, and in particular orthonormal, systems of n elements, but there are no orthogonal systems of $(n+1)$ elements. A system of n orthogonal vectors in E_n forms a basis in E_n, called an *orthogonal basis.*

THEOREM 2. *If vectors $e'_1, e'_2, \ldots, e'_n$ form an orthonormal system in E_n, any vector X of E_n is expressible (uniquely) in the form*

$$X = \sum_{i=1}^{n} x'_i e'_i, \qquad x'_i = Xe'_i, \tag{2.33}$$

$$\|X\|^2 = \sum_{i=1}^{n} x'^2_i. \tag{2.34}$$

The passage from unit vectors $e_1, e_2, \ldots, e_n$ in E_n to another orthonormal basis $e'_1, e'_2, \ldots, e'_n$ in E_n is called an *orthogonal transformation.* An inner product of vectors is unchanged in an orthogonal transformation, i.e. if

$$X = \sum_{i=1}^{k} x_i e_i = \sum_{i=1}^{n} x'_i e'_i, \quad Y = \sum_{i=1}^{n} y_i e_i = \sum_{i=1}^{n} y'_i e'_i.$$

then

$$XY = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x'_i y'_i.$$

In particular, the norms $\|X\| = \sqrt{XX}$ (see (2.34)) of all elements of E_n and the distance between any pair of such elements: $\varrho(X, Y) = \|X-Y\|$, are unchanged by an orthogonal transformation.

In general, if vectors $e'_1, e'_2, \ldots, e'_n$ form an orthogonal system, we have

$$X = \sum_{i=1}^{n} x'_i e'_i, \quad x'_i = \frac{Xe'_i}{\|e'_i\|^2}, \tag{2.33'}$$

$$\|X\|^2 = \sum_{i=1}^{n} x'^2_i \|e'_i\|^2 = \sum_{i=1}^{n} (Xe'_i)^2. \tag{2.34'}$$

In analogy with the theory of orthogonal series (see Chapter IV), the coefficients x'_i in equations (2.33) and (2.33′) can be termed *Fourier coefficients.*

8. Biorthogonal systems of vectors

Two vector systems $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ of E_n form a *biorthogonal system* if

$$X_iY_j = 0 \quad \text{for} \quad i \neq j, \quad X_iY_j \neq 0 \quad \text{for} \quad i = j. \tag{2.35}$$

Multiplying the vectors X_i or Y_i by constants, we can arrange that

$$X_iY_i = 1 \qquad (i = 1, 2, \ldots, n)$$

where the old notation is retained for the new vectors. In this case, every vector Z of E_n can be written as

$$Z = \sum_{i=1}^{n} x_iy_i, \quad x_i = ZX_i, \tag{2.36}$$

or as

$$Z = \sum_{i=1}^{n} y_iX_i, \quad y_i = ZY_i. \tag{2.36'}$$

Given a system of n independent vectors $X_1, X_2, \ldots, X_n$ in E_n, we can form a system of vectors $Y_1, Y_2, \ldots, Y_n$, such that the two systems $\{X_i\}$ and $\{Y_i\}$ are biorthogonal. If

$$X_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \quad \text{and} \quad Y_i = (y_{i1}, y_{i2}, \ldots, y_{in}),$$

formation of the system $\{Y_i\}$ from the system $\{X_i\}$ amounts to formation of the inverse $\|y_{ij}\|_{i,j}^{1,n}$ of the matrix $\|x_{ij}\|_{i,j}^{1,n}$. (See Chapter IV regarding orthogonal and biorthogonal systems of functions).

EXAMPLE 4. Let $X_i = \sum_{j=1}^{i} e_j (i = 1, 2, \ldots, n)$, where e_j are unit vectors in E_n; $Y_i = e_i - e_{i+1}$ $(i = 1, 2, \ldots, n-1)$, $Y_n = e_n$. In this case $\{X_i\}$ and $\{Y_i\}$ are biorthogonal systems in E_n.

9. The projection of a vector on to a manifold

Let E_k be a k-dimensional manifold in n-dimensional space E_n $(1 \leqq k < n)$. *The vector X of E_n is said to be orthogonal to E_k* (written as $X \perp E_k$) if X is orthogonal to any vector Y of E_k, i.e.

$$XY = 0 \quad \text{if} \quad Y \perp Ee.$$

A vector X_0 of E_k is called the *projection* of the vector X on to E_k if $X-X_0$ is orthogonal to E_k,

$$X_0 = \text{pr.}_{E_k} X. \tag{2.37}$$

We have:

$$X = X_0+(X-X_0), \quad (X-X_0) \perp E_k, \quad X_0 \in E_k.$$

when $X \in E_k$, to $\text{pr.}_{E_k} X = X$.

If $X \in E_k$, $\text{pr.}_{E_k} X = X$.

Any vector X of E_n has a unique projection on to E_k.

If X_0 is the projection of the vector X on to E_k, we have

$$(1) \quad ||X||^2 = ||X_0||^2+||X-X_0||^2, \tag{2.38}$$

$$(2) \quad ||X-Y||^2 \geqq ||X-X_0||^2, \tag{2.39}$$

where Y is any vector of E_k, and equality holds only when $Y = X_0$. In other words, the minimum of $||X-Y||^2$, where Y is any vector of E_k, is attained when

$$Y = X_0 = \text{pr.}_{E_k} X.$$

Let $U_0 \neq \theta$ be a vector of E_k. We shall write E_1 for the set of vectors of the form tU_0 $(-\infty < t < +\infty)$, i.e. the straight line $E_1 = tU_0$. *The projection of the vector X on to the vector* U_0 is defined as the projection of X on to the straight line E_1, i.e. we have:

$$\text{pr.}_{U_0} X = \frac{(XU_0)U_0}{||U_0||^2} \tag{2.40}$$

and in particular, if $||U_0|| = 1$, then

$$\text{pr.}_{U_0} X = (XU_0)U_0 = (||X|| \cos \varphi)U_0,$$

where φ is the angle between the vectors X and U_0.

Expansion (2.33) of a vector X into a system of orthogonal vectors $e'_j (j = 1, 2, \ldots, n)$ implies the representation of X as the sum of its projections on to the e'_j.

Let $X_1, X_2, \ldots, X_k$ be orthonormal vectors in E_n $(k < n)$; they form an orthogonal basis in E_k – their linear envelope. We can supplement them by the vectors X_{k+j} $(j = 1, 2, \ldots, n-k)$ such that the n vectors X_i $(i = 1, 2, \ldots, n)$ form an orthogonal basis in E_n.

Let X be any vector in E_n; then

$$\text{pr.}_{E_k} X = \sum_{i=1}^{k} c_i X_i, \quad c_i = XX_i.$$

The sum of the first k terms in expansion (2.33) *or* (2.33′) *is the projection of the vector X on to the linear envelope of the first k vectors X_i ($i = 1, 2, \ldots, n$).*

Let the vectors $\{X_i\}$ ($i = 1, 2, \ldots, k < n$) be orthogonal. Let us form all possible sums $\sum_{i=1}^{n} c_i e_i'$. The expression $\left\| X - \sum_{i=1}^{k} c_i e_i \right\|^2$ attains a minimum when $c_i = x_i'$, where c_i is the "Fourier coefficient"; $c_i = XX_i$ for $\|X_i\| = 1$, while in the general case $c_i = XX_i / \|X_i\|$,

THEOREM 3. *Let $\{L_i\}$ ($i = 1, 2, \ldots, k$) form a basis (generally speaking, not orthogonal in E_k), and X be any vector of E_n; then*

$$\text{pr.}_{E_k} X = \sum_{j=1}^{k} d_j L_j, \tag{2.41}$$

where d_j are given by the system of linear equations

$$\sum_{j=1}^{k} (L_i L_j) d_j = XL_i \cdot \qquad (i = 1, 2, \ldots, k), \tag{2.42}$$

the determinant of which is the same as the Gram determinant Γ. When $k = n$, the system (2.42) *becomes a system for determining the components of the vector X with respect to the basis $\{L_i\}$ ($i = 1, 2, \ldots, n$).*

§ 2. Passage to the limit, continuous functions and operators

1. Passage to the limit in n-dimensional space

Let L_n be a linear system with a chosen basis. Every element $X \in L_n$ is defined by its coordinates $x_1, x_2, \ldots, x_n$, $X = (x_1, x_2, \ldots, x_n)$. Let $\{X_n\} = \{X_1, X_2, \ldots\}$ be a sequence of elements of L_n, where $X_m = (x_{m1}, x_{m2}, \ldots, x_{mn})$. The element $X = (x_1, x_2, \ldots, x_n)$ of L_n is said to be *the limit of the sequence $\{X_m\}$* if

$$x_i = \lim_{m \to \infty} x_{mi} \tag{2.43}$$

(i.e. if all the coordinates of X are the limits of the corresponding coordinates of the terms of the sequence $\{X_m\}$). We write conventionally in this case

$$X = \lim_{m \to \infty} X_m, \tag{2.44}$$

and also speak of the sequence $\{X_m\}$ *converging to* X.

The coordinates of the elements $\{X_m\}$ and X depend on the choice of basis in L_n, but equation (2.43), i.e. (2.44) also, does not depend on the choice of basis in L_n: if the coordinates of X_m tend to the corresponding coordinates of X in one basis as $m \to \infty$, this property will hold for any choice of basis.

An n-dimensional linear system, in which passage to the limit is defined in the sense indicated, is called an *n-dimensional linear space.*

In Euclidean, and more generally, in normed (see § 3, sec. 3) spaces, where the distance between two points, or the norm of a vector, is defined, passage to the limit (2.44) can be defined as follows: $X = \lim\limits_{m \to \infty} X_m$ if

$$\lim_{m \to \infty} \varrho(X_m, X) = \lim_{m \to \infty} \|X_m - X\| = 0. \tag{2.45}$$

Condition (2.45) *is equivalent to condition* (2.43). Thus, if we define the distance between two points or the norm of a vector (§ 1, sec. 1 or 7) in a linear space, and make use of the definition of limit in accordance with (2.45), no changes whatever are introduced into the meaning of passage to the limit in L_n.†

There will be no loss of generality if we consider in this chapter passage to the limit in Euclidean spaces E_n.

As in the one-dimensional case (see Chapter I, § 3, sec. 2), we shall call a sequence $\{X_m\}$ of E_n *fundamental* if, given any $\varepsilon > 0$, here exists N such that

$$\|X_{m'} - X_m\| < \varepsilon,$$

for all $m' > N$, $m > N$.

A necessary and sufficient condition for a sequence $\{X_m\}$ *to have a limit is that it be fundamental.*

The concept of *limit point* (*point of condensation*) for a set $M \subset E_n$ is an immediate generalization of this concept for E_1 : X is called a

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† This introduction of a distance is called *metrization* of L_n.

limit point of the set $M \subset E_n$ if it is the limit of some sequence $\{X_m\}$ of points of M different from X.

An equivalent definition is: X is called a *limit point of the set* M if any sphere $S(X, \varepsilon)$ with centre at X contains a point of M different from X.

An analogue of the Bolzano–Weierstrass Theorem holds (see Chapter I, § 3, sec. 1).

Every bounded infinite set in E_n *has at least one limit point.*

An *open set* in E_n is a set U of points of E_n such that, if a point X belongs to U, there exists a sphere with centre at X contained in U. An open set is called a *domain* if any two points of it can be joined by a step-line, contained wholly in this set.

Examples of a domain are as follows: on a straight line, a finite or infinite interval; on a plane, the interior of a circle, triangle, strip between two parallel straight lines, etc.; in three-dimensional space, the interior of a sphere, parallelepiped, cone, etc.

An example of a domain in E_n is a parallelepiped (a parallelepiped in E_1 is an interval), i.e. a set of points $X(x_1, x_2, \ldots, x_n)$, the coordinates of which satisfy the inequalities $a_i < x_i < b_i$ $(i = 1, 2, \ldots, n)$, where a_i and b_i are given numbers.

The intersection of any finite number of domains is a domain.

A *boundary point of a domain* Q in E_n is a limit point of Q that does not belong to Q. The set of boundary points of Q is called the *boundary* of Q. A domain Q together with its boundary forms a *closed domain* (or *body*) in E_n.

For example, the boundary of an interval (a, b) consists of the points a and b, while the segment $[a, b]$ is a closed domain; the boundary of a sphere $S(X_0, r)$ is the surface of the sphere, i.e. the set of points X for which

$$\varrho(X, X_0) = \sqrt{\sum_{i=1}^{n} (x_i - x_0)^2} = r.$$

A *closed set* in E_n is a set in E_n containing all its limit points; for instance, closed domains are closed sets.

THEOREM 4. *The complement in* E_n *of an open set (domain) is a closed set, and of a closed set an open set.*

For example, the complement of the sphere $(\|X\| < r)$ is the closed set $(\|X\| \geqq r)$, the complement of the closed sphere $(\|X\| \leqq r)$ is the domain $(\|X\| > r)$.

2. Series of vectors

The laws for passage to the limits of vectors (points) of n-dimensional space E_n are essentially the same as the laws of passage to the limit in E_1; the generalization to the n-dimensional case of the theory of conditionally convergent series (i.e. series which are themselves convergent, while the series formed from the absolute values of their terms are divergent) is of some importance.

A vectot series in E_n:

$$\sum_{k=1}^{\infty} X_k = X_1+X_2+X_3+ \dots \tag{2.46}$$

is defined like a numerical series.

If $X_k = (x_{k1}, x_{k2}, \dots, x_{kn})$, series (2.46) corresponds to the k numerical series

$$\sum_{k=1}^{\infty} x_{ki} = x_{1i}+x_{2i}+x_{3i}+ \dots \qquad (i = 1, 2, \dots, n), \tag{2.47}$$

the terms of which are the coordinates of vectors X_k, i.e. of the terms of series (2.46).

The *partial sum* S_m of series (2.46) is the sum of its first m terms:

$$S_m = \sum_{k=1}^{m} X_k = X_1+X_2+ \dots +X_m.$$

Series (2.46) is said to be *convergent* if its partial sums S_m are convergent as $m \to \infty$ to some vector S, the sum of (2.46):

$$S = \sum_{k=1}^{\infty} X_k = \lim_{m\to\infty} S_m.$$

The convergence of series (2.46) is equivalent to the convergence of all the numerical series (2.47). If $S = (s_1, s_2, \dots, s_n)$, then

$$s_i = \sum_{k=1}^{\infty} x_{ki} \qquad (i = 1, 2, 3, \dots, n).$$

Series (2.46) is described as *absolutely convergent* if the series of the norms of its terms is convergent:

$$\sum_{k=1}^{\infty} ||X_k||.$$

The absolute convergence of series (2.46) is equivalent to the absolute convergence of the numerical series (2.47).

Series (2.46) is said to be *conditionally convergent* if it is convergent non-absolutely. We have Steinitz's Theorem, which is a generalization of Riemann's Theorem (see Chapter III, § 1, sec. 2).

(1) *If series* (2.46) *is absolutely convergent, its sum remains unaltered whenever the order of the terms is changed.*

(2) *If series* (2.46) *is conditionally convergent, its sum can be altered or it can be made divergent by changing the order of the terms.*

(3) *The sums of the series obtained by interchanging the terms of a convergent series* (2.46) *entirely fill some k-dimensional plane,* $0 \leqq k \leqq n$ (see § 1, sec. 5).

For an absolutely convergent series, $k = 0$; for a conditionally convergent series, $k > 0$.

EXAMPLE 5. Let $x_k = (-1)^k(1/k)$, and let s be an integer, $1 \leqq s \leqq n$. Every positive integer m can be written in the form $m = rs+k$, where r is an integer, $1 \leqq k \leqq s$. Let $X_m = X_{rs+k} = x_r e_k$, where e_k $(k = 1, 2, \ldots, s)$ are unit vectors.

By changing the order in the series $\sum\limits_m X_m$, we can now obtain as the sum, any vector of an s-dimensional manifold E_s — the linear envelope of the unit vectors $e_1, e_2, \ldots, e_s$ (see § 1, sec. 6).

3. Continuous functions of n variables

Functions of a point (vector) of n-dimensional space. Functions of n variables may naturally be treated as functions of a point (or vector) of n-dimensional space. Let each point X $(x_1, x_2, \ldots, x_n)$ of a set M in n-dimensional space E_n be associated with a number $f(X) = f(x_1, x_2, \ldots, x_n)$; we now say that a *function* $f(X) = f(x_1, x_2, \ldots, x_n)$ *of a point* (*vector*) X or a *function of n variables* — the coordinates of this point (vector) — is defined on the set M in E_n.

The concept of passage to the limit is introduced for functions in E_n as for functions of one variable.

Let $f(X)$ be a function defined on a set $M \subset E_n$ and let A be a limit point of M. We say that $f(X)$ *tends* to a number a when $X \in M$ tends to A, if

$$\lim_{n\to\infty} f(X_n) = a. \tag{2.48}$$

for any sequence $\{X_n\}$ of M, convergent to A.

This is written as

$$a = \lim_{X\to A(X\in M)} f(X). \tag{2.49}$$

(If M is a domain and A an interior point, it has to be mentioned that X_n tends to A while remaining in M.)

There is another definition of such a limit, equivalent to the above: (2.49) means that, given any $\varepsilon > 0$, there is a corresponding $\eta > 0$ such that, when

$$0 < \|X-A\| < \eta \quad \text{and} \quad X \in M$$

we have

$$|f(X)-a| < \varepsilon. \tag{2.50}$$

Continuous functions. We take a function $f(X) = f(x_1, x_2, \ldots, x_n)$ defined on a set $M \subset E_n$, and a point $\bar{X}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ of this set. We say that $f(X)$ is *continuous at the point* $\bar{X} \in M$ if

$$\lim_{m\to\infty,\ X_m\in M} f(X_m) = f(\bar{X}). \tag{2.51}$$

This definition is equivalent to the following: a function $f(X)$ is *continuous at a point* $\bar{X}$ of a set M if, given any $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon)$ such that, for all points $X \in M$ lying in the sphere

$$\varrho(X, \bar{X}) = \|X-\bar{X}\| < \eta, \tag{2.52}$$

we have

$$|f(X)-f(\bar{X})| < \varepsilon. \tag{2.52'}$$

A function that is continuous at every point of a set M is said to be *continuous in M*.

EXAMPLE 6. The function

$$\frac{1}{\sqrt{1-\|X\|^2}} = \frac{1}{\sqrt{1-\sum_{i=1}^{n} x_i^2}}$$

is defined and continuous at every point of the sphere $\|X\|^2 < 1$.

A linear function $f(X)$ (see § 1, sec. 5) can be defined as a function continuous in E_n and satisfying the additive condition:

$$f(X_1+X_2) = f(X_1)+f(X_2).$$

Uniform continuity. A function $f(X) = f(x_1, x_2, \ldots, x_n)$, defined on a set $M \subset E_n$, is *uniformly continuous* there if, given any $\varepsilon > 0$, there exists $\eta > 0$ such that, for any pair of points $X_1, X_2 \in M$ for which

$$\varrho(X_1, X_2) = \|X_1-X_2\| < \eta, \tag{2.53}$$

we have

$$|f(X_1)-f(X_2)| < \varepsilon. \tag{2.53'}$$

The number η in (2.52) depends on ε and on the choice of the point X, whereas in (2.53), defining the uniform continuity of a function, it depends only on ε and not on the choice of the points X_1 and X_2.

Every function, uniformly continuous on M, is continuous on M. The converse is not generally true (for example, the function $1/\sqrt{1-\|X\|^2}$, continuous inside the sphere $\|X\| < 1$, is uniformly continuous there).

As in the one-dimensional case (see Chapter I, § 3, sec. 11), we have

THEOREM 5. *A function $f(X) = f(x_1, x_2, \ldots, x_n)$, continuous on a bounded closed set U, is uniformly continuous there.*

Maximum and minimum. A function $f(X)$, defined on a set M, is said to be *bounded from above (below)* on M if there exists a constant C such that, for any point $X \in M$,

$$f(X) \leqq C \quad (f(X) \geqq C). \tag{2.54}$$

A function is bounded in M if it is bounded from above and below on M.

If $f(x)$ is bounded from above (below) on M, a number $C(c)$ exists which is the *strict upper* (*lower*) *bound* of the numbers $f(X)$, i.e. of the values of the function $f(X)$ at the points M:

$$C = \sup_{X \in M} f(X) \qquad \left(c = \inf_{X \in M} f(X)\right). \tag{2.55}$$

If there exists in M a point X_0 at which

$$f(X_0) = C = \sup_{X \in M} f(X),$$

the upper bound C is called the *maximum* of $f(X)$ on M:

$$C = f(X_0) = \max_{X \in M} f(X). \tag{2.56}$$

X_0 is called the *maximum point* for $f(X)$ on M.

Similarly, if there exists in M a point X_1 at which

$$f(X_1) = c,$$

twhere

$$c = \inf_{X \in M} f(X),$$

the lower bound c is called the *minimum* of $f(X)$ on M, while X_1 is the *minimum point* for $f(X)$ on M:

$$c = f(X_1) = \min_{X \in M} f(X). \tag{2.57}$$

We say in this case that $f(X)$ attains its maximum (minimum) on M at the point X_0 (X_1).

THEOREM 6. *A function continuous on a closed bounded set attains its maximum and minimum in the set.*

Jump functions. We often encounter in problems of mathematical physics the following generalization of the concept of jump to a function of n variables. Let Q_i be a domain in E_n. We say that it is "*inside*" its boundary Γ, while the domain $Q_e = E_n-(Qi+\Gamma)$ is "*outside*" the boundary Γ. We shall write A_i and A_e for the points of Q_i and Q_e respectively.† Let a function f (which may or may not be defined on Γ) be given in Q_i+Q_e. Suppose that the limit of $f(A_e)$ $[f(A_i)]$ exists, when A_e (A_i) tends to the point A of the boundary Γ; we shall write here:

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† i and e are the first letters of "*interior*" and "*exterior*".

$$f_e(A) = \lim f(A_e).$$

If $f_i(A) \neq f_e(A)$, we say that the function has a *jump*

$$f_e(A)-f_i(A)$$

on passing through the boundary Γ at the point A from Q_i to Q_e.

EXAMPLE 7. Let us take a bounded closed convex domain Q_i in E_2 with a boundary Γ (a smooth curve), and the domain $Q_e = E_2-Q_i-\Gamma$. Given an arbitrary point $A \in E_2$, we define a number $\alpha(A) > 0$, measuring the angle under which the curve Γ is seen from A. (It is formed by the rays joining A to all the points of Γ).

For points $A_i \in Q_i$, we have $\alpha(A_i) = 2\pi$; for points $A_e \in Q_e$, we have $0 < \alpha(A_e) < \pi$, where $\alpha(A_e) \to \pi$ if A_e tends to a point $A \in \Gamma$. We have $\alpha(A) = \pi$ for points A of the boundary Γ.

For the function $f(A) = \alpha(A)$ at the point $A \in \Gamma$, we have $f_e(A) = f(A) = \pi < f_i(A) = 2\pi$; the jump $f_e(A)-f_i(A) = -\pi$.

Functions depending on a parameter. Every function $\varphi(x_1, x_2, \ldots, x_m; t_1, t_2, \ldots, t_n)$ can be regarded as a family

$$\{f_{t_1t_2\ldots t_n}(x_1, x_2, \ldots, x_m) = \varphi(x_1, x_2, \ldots, x_m; t_1, t_2, \ldots, t_n)\}$$

of functions of m variables $x_1, x_2, \ldots, x_m$, each of which is defined by a set of n values of another group of variables $t_1, t_2, \ldots, t_n$ — called "*parameters*". We shall confine ourselves to the case of a system $\{f_t(x) = \varphi(x, t)\}$ of functions of one variable x, depending on one parameter t; t takes any value from a set T of the numerical axis E_1, while every function $f_t(x)$ is defined for x of the set $X \in E_1$ (the general case of n parameters and m variables is investigated similarly). Let t_0 be a limit-point of T. If, given any t and $x \in X$, the function $f_t(x)$ tends to $f(x)$ as $t \to t_0$ $(t \in T)$, we say that the functions $f_t(x)$ are *convergent on the set* X to the function $f(x)$. Given any $x \in X$ and any number $\eta > 0$, we can find a number ε, depending on η and x, $\varepsilon = \varepsilon(\eta, x)$, such that

$$|f_t(x)-f(x)| < \eta \tag{2.58}$$

for $0 < |t-t_0| < \varepsilon$ $(t \in T)$.

For example, let T be the interval $(0, \infty)$, and $X = [0, 1]$; the family of functions $\{f_t(x)\}$ is convergent to $f(x)$ on $[0, 1]$ as $t \to 0$ if

$$\left\{\begin{array}{l} \text{(a)}\ \ f_t(x) = \dfrac{\sin tx}{t}, \quad f(x) = x; \\ \text{(b)}\ \ f_t(x) = (1+tx)^{1/t}, \quad f(x) = e^x; \\ \text{(c)}\ \ f_t(x) = x^{1/t}, \quad f(x) = \begin{cases} 0 & \text{for} \quad 0 < x < 1, \\ 1 & \text{for} \quad x = 1. \end{cases} \end{array}\right.$$

The convergence of $\{f_t(x)\}$ to $f(x)$ on X is said to be *uniform* as $i \to t_0$ $(t \in T)$ if, given any $\eta > 0$, there exists $\varepsilon = \varepsilon(\eta)$, depending on η, such that (2.58) is satisfied for all $x \in X$ when $|t-t_0| < \varepsilon$ (ε no longer depends on the choice of x). In examples (a), (b), we have the case of uniform convergence, and of non-uniform in example (c).

The following are properties of the uniform convergence of a family of functions $\{f_t(x)\}$:

1°. *If the family* $\{f_t(x)\}$ *is uniformly convergent on* X *to* $f(x)$ *as* $t \to t_0$ $(t \in T)$, *and* $\{t_n\}$ *is any sequence of* T, *convergent to* t_0, *the sequence of functions* $\{f_{t_n}(x)\}$ *is uniformly convergent to* $f(x)$.

2°. *If all the functions* $f_t(x)$ *are continuous on* X, *their uniform limit is a function* $f(x)$ *continuous on* X. *(See examples* (a), (b). *In example* (c), *the limit function is discontinuous, which points to the non-uniform convergence of* $\{f_t(x)\}$ *to* $f(x)$.*)*

3°. *Let* $\varphi(x, t)$ *be a function of two variables continuous in the rectangle* $a \leqq x \leqq a_1$, $b \leqq t \leqq b_1$. *On writing* $f_t(x) = \varphi(x, t)$ *for* $t \in [b, b_1)$, $f(x) = \varphi(x, b_1)$, *we find that the functions* $f_t(x)$ *are uniformly convergent to* $f(x)$ *as* $t \to b_1$.

4. Periodic functions of n variables. Manifolds of constancy

Let E_k $(1 \leqq k \leqq n)$ be a linear manifold in E_n, and $f(X)$ a function defined in E_n. We say that E_k is a *manifold of constancy* for the function $f(X)$ if, given any $X \in E_n$ and any Y of E_k, we have

$$f(X+Y) = f(X). \tag{2.59}$$

For example, in the case of the function $f(x+y)$ of two variables x and y, the straight line $x = -y$ is a manifold of constancy. For, if we add to the vector (x, y) any vector $(x_0, -x_0)$ of this straight line, the function is unchanged.

Let us bring in the basis in E_n of elements $Y_1, Y_2, \ldots, Y_k; Z_1, Z_2, \ldots \ldots, Z_{n-k}$, where $Y_1, Y_2, \ldots, Y_k$ form a basis in the manifold of constancy E_k. Every element $X \in E_n$ can be written in the form (see (2.29))

$$\left.\begin{aligned} X &= Y+Z, \\ Y \in E_k, \quad Z &= \sum_{i=1}^{n-k} z_i Z_i, \end{aligned}\right\}$$

where

$$f(X) = f(Y+Z) = f(Z).$$

The function $f(X) = f(Z)$ reduces to a function of $n-k$ variables — the coordinates $Z_1, Z_2, \ldots, Z_{n-k}$ of the vector Z.

A period of a function of n variables. A *period* of a function $f(X)$, defined on E_n, is vector ω (different from the zero θ) such that, for any X,

$$f(X+\omega) = f(X). \tag{2.60}$$

It follows from (2.60) that *all the elements $Y \neq \theta$ of a manifold of constancy of a function f are periods of f.* A function $f(X)$, having a period that does not belong to the manifold of constancy, is described as *periodic.*

We shall confine ourselves to periodic functions that have no manifolds on constancy. Notice that *every periodic function has an infinite set of periods* (in addition to ω, every vector of the form $k\omega$, where k is any integer, is a period).

THEOREM 7. *If ω and ω_1 are periods of a function $f(X)$, their sum $\omega+\omega_1$, is also a period.*

If $\omega_1, \omega_2, \ldots, \omega_m$ are periods of $f(X)$, any vector ω of the form

$$\omega = \sum_{i=1}^{m} n_i \omega_i,$$

where n_i are arbitrary integers, is also a period.

The periods $\omega_1, \omega_2, \ldots, \omega_m$ form a *system of fundamental periods* of $f(X)$ if all the periods of $f(X)$ are expressible as linear combinations with integral coefficients of these periods and are not expressible as such linear combinations of a smaller number of periods.

Example 8. Let us take $f(X) = f(x_1, x_2) = \sin x_1 \cos x_2$ on the plane E_2. It has a system of two fundamental periods, namely, the vectors ω_1 $(2\pi, 0)$ and ω_2 $(0, 2\pi)$. We could also take as fundamental periods ω_1 and $\omega_3 = (2\pi, 2\pi)$ (in which case $\omega_2 = \omega_3 - \omega_1$).

Theorem 8. *A continuous periodic function of n variables, having no manifold of constancy, has a fundamental system of periods consisting of not more than n periods.*

Example 9. Let $\omega_1, \omega_2, \dots, \omega_n$ be a system of linearly independent vectors in E_n. Let us form the set A of all vectors (points) of the form

$$\sum_{i=1}^{n} n_i \omega_i,$$

where the n_i are integers. (Such a set is called a *net of integers* in E_n.) We shall write $\varrho(X, A)$ for the distance from the point $X \in E_n$ to the nearest point of A; $\varrho(X, A)$ is a function of X. It has a system of fundamental periods consisting of the vectors $\omega_1, \omega_2, \dots, \omega_n$.

Example 10. A continuous function of a complex variable $f(x+iy) = P(x, y) + iQ(x, y)$ (not equal to a constant) cannot have more than two independent periods (Jacobi's theorem).

5. Passage to the limit for linear envelopes

Let $a_1, a_2, \dots, a_p$ be linearly independent elements in E. We shall write $L(a_1, a_2, \dots, a_p)$ for their linear envelope. If $b_1, b_2, \dots, b_p$ is another basis in the p-dimensional manifold $L(a_1, \dots, a_p)$, then $L_p(b_1, b_2, \dots, b_p) = L_p(a_1, a_2, \dots, a_p)$.

Let us consider a family of vectors l_α, continuously dependent on a parameter α, such that, when $\alpha_2 \to \alpha$, $\alpha_1 \to \alpha (\alpha_2 \neq \alpha_1)$, the expression $(l_{\alpha_2} - l_{\alpha_1})/(\alpha_2 - \alpha_1)$ tends to a vector which we shall denote by $dl_\alpha/d\alpha$. We shall assume that l_{α_1}, and l_{α_2} are linearly independent when $\alpha_2 \neq \alpha_1$. Having chosen another basis in $L_2(l_{\alpha_1}, l_{\alpha_2})$:

$$l_{\alpha_1}, \ \frac{l_{\alpha_2} - l_{\alpha_1}}{\alpha_2 - \alpha_1},$$

we have

$$L_2(l_{\alpha_1}, l_{\alpha_2}) = L_2\left(l_{\alpha_1}, \frac{l_{\alpha_2} - l_{\alpha_1}}{\alpha_2 - \alpha_1}\right).$$

Assuming that the vectors l_α and $dl_\alpha/d\alpha$ are linearly independent, we find that, as $\alpha_1 \to \alpha$, $\alpha_2 \to \alpha$, then the plane $L_2(l_{\alpha_1}, l_{\alpha_2}) = L_2(l_{\alpha_1}, (l_{\alpha_2}-l_{\alpha_1})/(\alpha_2-\alpha_2))$ tends to the plane $L_2(l_\alpha, dl_\alpha/d\alpha)$.

Given similar conditions (assuming that the systems of vectors $l_{\alpha_1}, l_{\alpha_2}, \ldots, l_{\alpha_p}$, where no two of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_p$ are equal, and of the vectors $l_\alpha, dl_\alpha/d\alpha, \ldots, d^pl_\alpha/d\alpha^p$, are linearly independent), we find that, as $\alpha_i \to \alpha$ ($i = 1, 2, \ldots, p$):

$$L_p(l_{\alpha_1}, l_{\alpha_2}, \ldots, l_{\alpha_p}) \to L_p\left(l_\alpha, \frac{d}{d\alpha} l_\alpha, \ldots, \frac{d^{p-1}}{dx^{p-1}} l_\alpha\right).$$

Given a family of functions $f_\alpha(x)$, dependent on a parameter α, with the same conditions and with $\alpha_i \to \alpha(i = 1, 2, \ldots, p)$, we find that $L_p(f_{\alpha_1}(x), f_{\alpha_2}(x), \ldots, f_{\alpha_p}(x))$ tends to $L_p(f_\alpha(x), \partial f_\alpha(x)/\partial\alpha, \ldots, \partial^{p-1} f_\alpha(x)/\partial\alpha^{p-1})$. For example, as $\alpha_i \to \alpha$ $(i = 1, 2, \ldots)$, we have

$$\left.\begin{aligned}
&L_p(e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_p x}) \to L_p(e^{\alpha x}, xe^{\alpha x}, \ldots, x^{p-1}e^{\alpha x}),\\
&L_p\left(\frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2}, \ldots, \frac{1}{x-\alpha_p}\right) \to\\
&\qquad\qquad \to L_p\left(\frac{1}{x-\alpha}, \frac{1}{(x-\alpha)^2}, \ldots, \frac{1}{(x-\alpha)^p}\right),\\
&L_p(x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_p}) \to L_p(\lambda^\alpha, x^\alpha \ln x, \ldots, x^\alpha \ln^{n-1} x),\\
&L_2(|x-\alpha_1|, |x-\alpha_2|) \to L_2(|x-\alpha|, \operatorname{sign}|x-\alpha|).
\end{aligned}\right\}$$

Let A_α be an n-dimensional matrix, $\lambda_{i\alpha}$ $(i = 1, 2, \ldots, n)$ be its eigenvalues (simple for $\alpha \neq 0$), $x_{i\alpha}$ the corresponding eigenvectors: $A_\alpha x_{i\alpha} = \lambda_{i\alpha} x_{i\alpha}$, $\|x_{i\alpha}\| = 1$. Suppose that k of the numbers $\lambda_{i\alpha}$ $(i = 1, 2, \ldots, k \leq n)$ tend as $\alpha \to 0$ to the eigenvalue λ_0 of the matrix A_0, while the eigenvectors x_i $(i = 1, 2, \ldots, k)$ tend to the eigenvector x_1 of the matrix A_0 corresponding to the eigenvalue λ_0. The linear envelopes $L_k(x_1, x_{2\alpha}, \ldots, x_k)$ now tend to the linear envelope $L_k(x_1, x_2, \ldots, x_k)$ where x_i with $i > 1$ are augmented eigenvectors of the matrix A_0 : $A_0 x_i - \lambda_0 x_i = x_{i-1}$ $(i = 2, 3, \ldots, k)$.

6. Operators from E_n into E_m

Let Q be a set in E_n (which can coincide with E_n itself); we associate with every element $X(x_1, x_2, \ldots, x_n)$ of Q an element $Y(y_1, y_2, \ldots, y_m) = f(X)$ of E_m. We say in this case that *an operator f is given* from E_n into E_m, Q being called the *domain of definition* of the operator. We can also say that we have a *mapping* f of the set Q of E_n into E_m or that the operator f *performs a mapping* of the set Q from the space E_n into the space E_m.

If

$$Y = (y_1, y_2, \dots, y_m), \quad X = (x_1, x_2, \dots, x_n),$$

$$Y = f(X), \tag{2.61}$$

then

$$y_i = f_i(X) = f_i(x_1, x_2, \dots, x_n), \tag{2.62}$$

where f_i is a function of n variables defined in Q, formula (2.62) being the coordinate form of writing the operator f, and (2.61) the vector form. The functions f_i $(i = 1, 2, \dots, m)$ are called the *components* of the operator f, and we write

$$f = (f_1, f_2, \dots, f_m).$$

Every set of m functions $f_i(x_1, x_2, \dots, x_n)$ $(i = 1, 2, \dots, m)$ of n variables defines an operator f from E_n into E_m, where $f = (f_1, f_2, \dots, f_m)$.

An ordinary function $f(x_1, x_2, \dots, x_n)$ is an operator from E_n into E_1 (into the numerical axis).

An operator $f(X)$ is said to be *continuous* in Q if

$$X_n \xrightarrow[n\to\infty]{} X$$

X_n and X belong to Q) implies that

$$f(X_n) \xrightarrow[n\to\infty]{} f(X).$$

The necessary and sufficient condition for the operator $f(X) = (f_1(X), f_2(X), \dots, f_n(X))$ *to be continuous is that all the functions* $f_i(X) = f_i(x_1, x_2, \dots, x_n)$ *be continuous in* Q.

For example, the functions

$$\left.\begin{aligned} Z_1 &= \sqrt{1-x_1^2-x_2^2-x_3^2} = \sqrt{1-\|X\|^2}, \\ Z_2 &= x_1+x_2+x_3 \end{aligned}\right\}$$

yield an operator from $E_3(x_1, x_2, x_3)$ into the plane $E_2(Z_1, Z_2)$, defined in the three-dimensional sphere $\|X\|^2 < 1$ and continuous there. Functions of a complex variable provide us with examples of operators from E_2 into E_2 (mapping of a plane into a plane).

The term *vector function* is occasionally used instead of operator from E_n into E_m. An ordinary function of n variables ($n > 1$, $m = 1$) is called a *scalar function of a vector variable;* an operator from E_1 into E_n where $n > 1$, is called a *vector function of a scalar variable* or simply a *vector function,* and an operator from E_n into E_m, where $n > 1$, $m > 1$, is called a *vector function of a vector variable.*

Let $X = X(t)$ be a given vector function, where t runs over the numerical axis (or a segment of it), and let X be a point of E_m. The set of points of E_m of the form $X(t)$ is called a *curve* or *line* in E_m.

Linear operators. A *linear operator* (*mapping*) from $E_n(x_1, x_2, \ldots, x_n)$ into $E_m(y_1, y_2, \ldots, y_m)$ is an operator $Y(X)$ satisfying the conditions:

(1) $Y(X_1+X_2) = Y(X_1)+Y(X_2)$;

(2) the operator $Y(X)$ is continuous.

It follows from conditions (1) and (2) that a linear operator is *homogeneous:*

$$Y(\lambda X) = \lambda Y(X).$$

THEOREM 9. *A linear operator* $Y(X)$, *where* $Y = (y_1, y_2, \ldots, y_m)$, $X = (x_1, x_2, \ldots, x_n)$, *has the form:*

$$y_i = a_{i1}x_1+a_{i2}x_2+ \ldots +a_{in}x_n \qquad (i = 1, \ldots, n), \tag{2.63}$$

i.e. every coordinate $y_i(X)$ *of the operator* $Y(X)$ *is a linear function of* $X = (x_1, x_2, \ldots, x_n)$.

Every linear operator $Y(X)$ from E_n into E_m is defined by a rectangular matrix

$$\left.\begin{array}{c} A = \|a_{ij}\| \quad (i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n), \\ Y = AX. \end{array}\right\} \tag{2.64}$$

Conversely, every such matrix defines a linear operator from E_n into E_m (in accordance with (2.63)). When $m = n$, *a linear operator from* E_n *into* E_n *is defined by a square matrix* $A = \|a_{ij}\|_{i,\,j}^{1,\,n}$

Formula (2.63) is the *coordinate form* of a linear operator (transformation) from E_n into E_m.

7. Iterative sequences

Let

$$\{X_n\} = \{X_0, X_1, X_2, \ldots\}, \quad X_n = (x_{n1}, x_{n2}, \ldots, x_{nk}) \tag{2.65}$$

be a sequence of vectors in E_k, the corresponding k numerical sequences being

$$\{x_{ni}\} = \{x_{0i}, x_{1i}, x_{2i}, \ldots\} \qquad (i = 1, 2, \ldots, k). \tag{2.66}$$

The vector sequence (2.65) (or system (2.66) of numerical sequences) is described as *iterative* if

$$X_n = f(X_{n-1}) \qquad (n = 1, 2, 3, \ldots), \tag{2.67}$$

where f is an operator from E_k into E_k. If $f = (f_1, f_2, \ldots, f_k)$, equation (2.67) may be written in the coordinate form as

$$x_{ni} = f_i(x_{n-1,1}\ x_{n-1,2}, \ldots, x_{n-1,k}) \qquad (i = 1, 2, \ldots, k). \tag{2.68}$$

On putting $X_0 = (x_{01}, x_{02}, \ldots, x_{0k})$, we can find successively from (2.67) or (2.68):

$$X_1 = (x_{11}, x_{12}, \ldots, x_{1k}), \quad X_2 = (x_{21}, x_{22}, \ldots, x_{2k}), \ldots$$

Relationship (2.68) is sometimes written as

$$x_{ni} = f_i(x_{n1}, x_{n2} \ldots, x_{n,i-1}, x_{n-1,i}, x_{n-1,i+1}, \ldots, x_{n-1,k}). \tag{2.69}$$

(When finding the ith component of the nth vector X_n, use is made of the components $x_{n1}, x_{n2}, \ldots, x_{n,i-1}$ already found.)

EXAMPLE 11. K. F. Gauss considered a sequence of pairs of positive numbers $\{\alpha_n, \beta_n\}$ $(n = 0, 1, 2, \ldots)$ or of plane vectors, where α_0, β_0 are given $(\alpha_0 \geqq \beta_0)$, and

$$\alpha_n = \frac{\alpha_{n-1}+\beta_{n-1}}{2}, \qquad \beta_n = \sqrt{\alpha_n\beta_n}.$$

As $n \to \infty$, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ tend to a common limit $a = \lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n$ – the *arithmetico-geometric mean* of the *numbers* α_0, β_0:

$$a = a(\alpha_0, \beta_0) = \frac{\pi}{2G}, \qquad G = \frac{1}{\alpha_0}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\dfrac{\alpha_0^2-\beta_0^2}{\alpha_0^2}\sin^2\varphi}}.$$

EXAMPLE 12. Let us take the sequence of partial sums $s_n = 1 + (x/1!) + \ldots + (x^n/n!)$ of the power series for e^x. On writing $\alpha_n = n$,

$\beta_n = (x^n/n!)$ we have: $(\alpha_0, \beta_0, s_0) = (0, 1, 1)$. We get an iterative sequence (α_n, β_n, s_n) of the form (2.69):

$$\alpha_n = \alpha_{n-1} + 1, \quad \beta_n = \beta_{n-1}\frac{x}{\alpha_n}, \quad s_n = s_{n-1} + \beta_n.$$

The formula provides a unified scheme for obtaining successively (α_n, β_n, s_n) (i.e. s_n also) for $n = 1, 2, 3, \ldots$. The unified passage from one term of an iterative sequence to the next term is very convenient for computation on programme-controlled machines.

Iterative processes. We can generalize the concepts of § 2, sec. 6, to operators from E_n into E_n.

Let f be a continuous operator from E_n into E_n:

$$Y = f(X), \quad Y = (y_1, y_2, \ldots, y_n), \quad X = (x_1, x_2, \ldots, x_n),$$

or, in the coordinate form:

$$y_i = f_i(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n).$$

Let us investigate the equation

$$X = f(X) \tag{2.70}$$

or, in the coordinate form, the system of equations

$$X_i = f_i(x_1, x_2, \ldots, x_n). \tag{2.71}$$

We form the iterative sequence of elements

$$X_0, X_1, X_2, \ldots, X_m, \ldots; \quad X_m = (x_{m1}, x_{m2}, \ldots, x_{mn}),$$

where $X_0(x_{01}, x_{02} \ldots, x_{0n})$ is an arbitrary element of E_n,

$$X_{m+1} = f(X_m) \qquad (m = 0, 1, 2, \ldots), \tag{2.72}$$

and in the coordinate form

$$x_{m+1, i} = f_i(x_{m1}, x_{m2}, \ldots, x_{mn}) \qquad (i = 1, 2, \ldots, n). \tag{2.73}$$

If the sequence X_m is convergent to X^*, X^* is a solution of equation (2.70).

The solution X^* is called a *fixed point of the transformation* $Y = f(X)$.

8. The principle of contraction mappings

We introduce a norm into the space E_n (see § 1, sec. 2). An operator (transformation) f from E_n into E_n is said to be a contraction mapping if there is a constant q $(0 < q < 1)$ such that, for any X, $X_1 \in E_n$,

$$\|f(X_1)-f(X)\| \leqq q\|X_1-X\|. \tag{2.74}$$

The following theorem provides a condition for the existence of a fixed point of the transformation $Y=f(X)$, i.e. of a solution of equation (2.70), and at the same time, a condition for convergence of the iterative process (2.72).

THEOREM 10. (the principle of contraction mappings). *If $Y=f(X)$ is a contraction mapping (i.e. condition* (2.74) *is satisfied with $q<1$), then:*

(1) *there exists a solution $X=X^*$ of equation* (2.70), *i.e. a fixed point X^* of the transformation $Y=f(X)$;*

(2) *this solution is unique;*

(3) *whatever the initial approximation X_0, the iterative process* (2.72) *is convergent to the solution X^*,*

$$X^* = \lim_{m\to\infty} X_m. \tag{2.75}$$

(4) *the sequence X_m is convergent to X^* as fast as the convergence of a geometric progression with ratio q, in fact,*

$$\|X^*-X_n\| \leqq \frac{q^n}{1-q}\|X_1-X_0\|. \tag{2.76}$$

We shall mention some sufficient conditions for an operator to be a contraction mapping under different norms. It is assumed that the functions $f_i(X)=f_i(x_1, x_2, \ldots, x_n)$ – the coordinates of the operator (X) – have continuous partial derivatives with respect to all the arguments.

For condition (2.74) *to be satisfied, (also the consequences of the theorem), it is sufficient that:*

(a) *in the Euclidean metric,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial f_i(x_1, x_2, \ldots, x_n)}{\partial x_j}\right]^2 \leqq q < 1; \tag{2.77}$$

(b) *in the metric* $m_{(n)}$ (see p. 97),

$$\sum_{j=1}^{n}\left|\frac{\partial f_i(x_1, x_2, \ldots, x_n)}{\partial x_j}\right| \leqq q_i < 1 \qquad (i = 1, 2, \ldots, n). \tag{2.77'}$$

Application to the solution of systems of linear algebraic equations. Let us take the system of linear algebraic equations

$$y_i = \sum_{j=1}^{n} a_{ij}x_j + b_i \tag{2.78}$$

or, in the vector form,

$$Y = AX + B, \tag{2.79}$$

$$A = \|a_{ij}\|, \quad B = (b_1, b_2, \ldots, b_n). \tag{2.80}$$

Given any initial vector $X_0(x_{01}, x_{02}, \ldots, x_{0n})$, we form a system of vectors $X_0, X_1, X_2, \ldots, X_m, \ldots$

$$X_m = (x_{m1}, x_{m2}, \ldots, x_{mn}), \tag{2.81}$$

where

$$X_{m+1} = AX_m + B \qquad (m = 0, 1, 2, \ldots) \tag{2.82}$$

or, in the coordinate form,

$$x_{m+1,i} = \sum_{j=1}^{n} a_{ij}x_{mj} + b_i \qquad (i = 1, 2, \ldots, n). \tag{2.83}$$

It follows from (2.77) and (2.77′) that a sufficient condition for convergence of the vector sequence X_m as $m \to \infty$ to the vector solution $X^*(x_1^*, x_2^*, \ldots, x_n^*)$ of system (2.79) or system (2.78) is that

$$\text{(a)} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = q < 1$$

or

$$\left.\begin{aligned} &\text{(b)} \quad \sum_{j=1}^{n} |a_{ij}| = q_i < 1 \qquad && i = 1, 2, \ldots, n) \\ & \qquad\qquad (q = \max q_i; && i = 1, 2, \ldots, n), \end{aligned}\right\}$$

the accuracy $\|X^* - X_n\|$ of the nth approximation X_n being defined by (2.76); in case (a) the norm is Euclidean, in case (b) it is $m_{(n)}$.

§3. Convex bodies in n-dimensional space

The theory of convex bodies in n-dimensional spaces was developed by G. Minkovskii; it has found applications in numerous problems of analysis, geometry and number theory, and more recently in new branches of applied mathematics: in the theory of games and in linear programming. This theory provided the basis for developing the theory of n-dimensional normed spaces, the infinite-dimensional generalization of which plays an important role in analysis.

1. Fundamental definitions

A *convex set* in space E_n is one which contains, in addition to two points A and B of the set, every segment joining these points.

Particular cases: a *convex domain* Q is a domain which is a convex set, a *convex body* $\bar{Q}$ is the convex domain Q together with its boundary Γ. Points of the domain Q are called *interior*, and points of Γ *boundary points* of Q.

On a straight line, an interval is a convex domain, and a segment a convex body; on a plane, the interior of a circle or triangle is a convex domain and the circle and triangle (including their boundaries) are convex bodies; in three-dimensional space, the interior of a cylinder, sphere, cube are convex domains, and the cylinder, sphere and cube (including their boundaries) are convex bodies, the sphere and cube being bounded convex solids, and the cylinder an infinite convex solid.

A closed convex set, lying in a k-dimensional plane and not lying in any (k—1)-dimensional plane, is called a *k-dimensional convex body* ($0 \leqq k \leqq n$). A point will be described as a *zero-dimensional convex body*. (The empty set is also regarded as convex.)

A convex body (set) may be bounded or unbounded (cf. the above examples). We can quote as an example of an unbounded convex body a *half-space* in E_n, i.e. the set of points X satisfying the inequality ($fX \leqq C$), where f is a linear functional in E_n, and C a constant; the whole of the space E_n and any hyperplane in it are examples of unbounded convex sets.

The intersection of convex sets in E_n is a convex set, the intersection of convex bodies in E_n is a convex k-dimensional body ($k \leqq n$).

A convex polyhedron is a closed convex body — the intersection of a finite number of half-spaces.

The *convex envelope of a set* M in E_n is the set Q of points $X \in E_n$ expressible in the form

$$X = \sum_{i=1}^{l} \lambda_i X_i, \tag{2.84}$$

where l is any integer, $l \leqq n$, X_i are arbitrary points of M, λ_i are arbitrary non-negative numbers satisfying the condition

$$\sum_{i=1}^{l} \lambda_i = 1.$$

EXAMPLE 13. The convex envelope of a pair of points A and B is the segment joining them; the convex envelope of three points not lying on a straight line is a triangle; the convex envelope of four points, not on the same plane, is a tetrahedron with vertices at these points.

The *convex envelope of a set* M is the intersection of all convex sets containing M, or the least convex set that contains M. Every bounded convex polyhedron is the convex envelope of a finite number of points – the vertices of the polyhedron.

Limit points. A point A is a *limit point* of a convex body Q if A is not an interior point of any segment belonging to Q.

Every limit point of Q is a *boundary point* of Q; but not every boundary point is a limit point. For example, only the vertices of a polyhedron (polygon) are its limit points. In the case of a circle and sphere, all the boundary points are limit points. In future, a convex body will be taken to mean a bounded convex body.

A convex body Q is the convex envelope of its limit points.

2. Convex functions

A function $f(X) = f(x_1, x_2, \ldots, x_n)$, defined in E_n (or on a convex set Q of E_n), is described as *convex* if, given any X and Y of E_n (of Q), we have

$$f\left(\frac{X+Y}{2}\right) \leqq \frac{1}{2}[f(X)+f(Y)] \tag{2.85}$$

(and *concave*, if the reverse inequality is satisfied).

A convex function also satisfies the more general inequality:

$$f[(1-t)X+tY] \leqq (1-t)f(X)+tf(Y) \qquad (2.85'$$

for any $t \in [0, 1]$.

(See Chapter I, § 3, sec. 17, for convex functions of one variable.)

If $f(X)$ is a convex function, c_0 is a lower bound in E_n (in Q), the set of points of E_n (of Q), satisfying the inequality

$$f(X) \leqq c,$$

with any $c \geqq c_0$, is a convex body (set).

EXAMPLE 14. On the plane $E_2\,(x_1, x_2)$, the functions

$$f_p(x_1, x_2) = (|x_1|^p+|x_2|^p)^{1/p}$$

with $p \geqq 1$ are convex functions. As $p \to \infty$, $f_p(x_1, x_2)$ tends to the function $f_\infty(x_1, x_2) = \max(|x_1|, |x_2|)$ (see Fig. 2). The sketch illustrates the curves $f_p(x_1, x_2) = 1$ for different p. They all pass through the points $A_1\,(1, 0)$, $A_2\,(0, 1)$, $A_3\,(-1, 0)$, $A_4\,(0, -1)$.

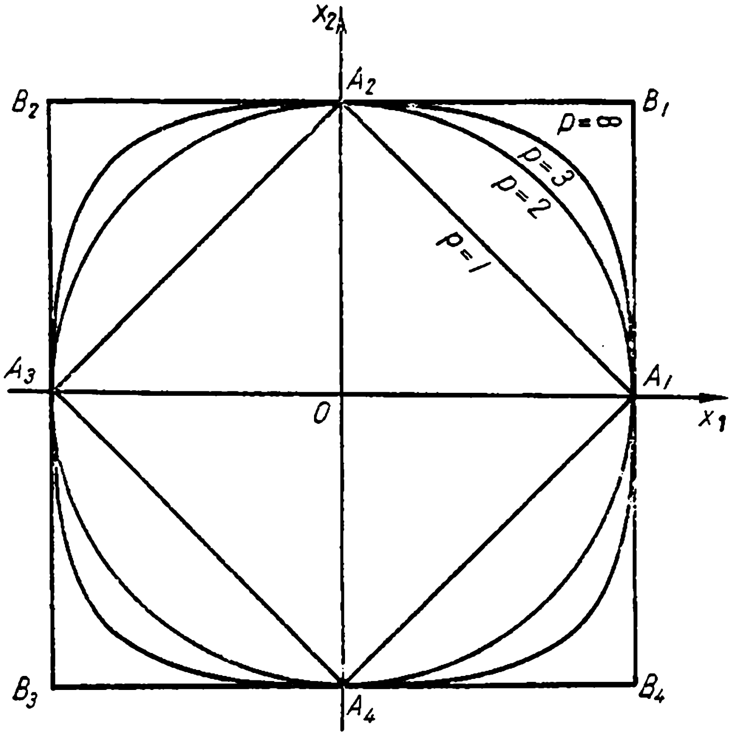

FIG. 2.

The curve $f_\infty(x_1, x_2) = 1$ is the boundary of the square with vertices A_1, A_2, A_3, A_4.

The curves $f_p = 1$, where $1 < p < \infty$, lie between these two curves.

The curve $f_2\,(x_1, x_2) = 1$ is the circle of radius 1.

The figures $f_p \leqq 1$, bounded by these curves, are convex. When $p<1$, such figures are no longer convex. For instance, when $p=2/3$, we get a figure bounded by an astroid.

3. Convex bodies and the norm of a vector

Let Q be a convex body in E_n, having θ as an interior point. Every point X of E_n can be written (uniquely, when $X \neq \theta$) as $X = \lambda X_0$, where $\lambda > 0$, X_0 is a point of the boundary of Q. If X lies outside Q, then $\lambda > 1$; if X lies inside Q, $\lambda < 1$; if X lies on the boundary of Q, $\lambda = 1$. The point θ can be written as $0.X_0$ (i.e. $\lambda = 0$ for the point θ).

We now define a function $\varphi_Q(X)$ in E_n as follows: at the point $X = \lambda X_0$,

$$\varphi_Q(X) = \lambda;$$

$\varphi_Q(X)$ is greater than, equal to, or less than 1, depending on whether X is outside, inside, or on the boundary of Q respectively,

$$\varphi_Q(\theta) = 0.$$

$\varphi_Q(X)$ *is a convex function.* The body Q is the set of points of E_n for which $\varphi_Q(X) \leqq 1$, while the boundary of Q is defined by the equation $\varphi_Q(X) = 1$.

$\varphi_Q(X)$ has the following properties:

(1) $\varphi_Q(X) \geqq 0$, where $\varphi_Q = 0$ only when $X = \theta$;

(2) When $\lambda \geqq 0$, $\varphi_Q(\lambda X) = \lambda \varphi_Q(X)$ (implying a positive homogeneous function);

(3) $$\varphi_Q(X+Y) \leqq \varphi_Q(X) + \varphi_Q(Y).$$

If Q is a central symmetric convex body with centre at θ (i.e. $X \in Q$ implies $-X \in Q$), we have the additional property:

$$\varphi_Q(\lambda X) = |\lambda| \varphi_Q(X) \quad \text{for} \quad \lambda < 0.$$

Property (2) becomes the stronger property:

(2′) $$\varphi_Q(\lambda X) = |\lambda| \varphi_Q(X)$$

for any real λ. If φ is a given function satisfying conditions (1)–(3), the set Q defined by $\varphi_Q(X) \leqq 1$ is a convex body, while it is a centrally symmetric convex body with centre at θ when condition (2′) is satisfied.

The Euclidean norm $||X||$ of the vector $X \in E_n$ satisfies conditions (1), (2′), (3). We can generalize the concept of norm of a vector in n-dimensional space: the *norm* can be taken as any function $\varphi(X)$ that satisfies conditions (1), (2), (3). We shall write $E_{n,\varphi}$ for such a space.

If $||X|| = \varphi(X)$, conditions (1) – (3) can be rewritten as:

(1) $||X|| \geqq 0$, and $||X|| = 0$ only when $X = 0$;

(2) When $\lambda > 0$, we have $||\lambda X|| = \lambda ||X||$;

(3) $||X+Y|| \leqq ||X||+||Y||$ (the *triangle inequality*).

When φ is even, condition (2) is replaced by the stronger one:

(2′) $||\lambda X|| = |\lambda|\ ||X||$ (for any λ).

Spaces E_n, in which a norm is introduced, satisfying the above conditions, are said to be *normed.* For instance, Euclidean spaces are normed (see § 1, sec. 7).

4. Support hyperplanes

Let us take the linear form $fX = \sum_{i=1}^{n} l_i x_i$ in E_n, the hyperplane $(fX = C)$ and the half-space $(fX \leqq C)$. We call $(fX = C)$ a *hyperplane of support for the convex body Q* if Q lies wholly in the half-space $(fX \leqq C)$ and the hyperplane has points in common with the boundary of Q.

EXAMPLE 15. In the two-dimensional case, the hyperplanes of support reduce to *lines of support.* In the case of a circle, the lines of support are the same as the tangents. In the case of a triangle, the lines of support are the three lines along which its sides lie, together with all the straight lines passing through its vertices and lying in its exterior angles. In three-dimensional space, hyperplanes of support become planes of support; for a sphere, the planes of support are its tangent planes, and for a cube, its boundary planes and the other planes that pass through the intersection of two sides, or merely through a vertex and do not cut the interior of the cube.

Let the equation of the hyperplane of support be

$$fX = C. \tag{2.86}$$

THEOREM 11. *The number C on the right-hand side of* (2.86) *is defined by the equation*

$$C = \max_{X \in Q} fX. \tag{2.87}$$

The equation of the hyperplane of support therefore has the form

$$fX = \max_{X \in Q} fX. \tag{2.88}$$

THEOREM 12. *The linear function fX attains its maximum C on Q at a limit point of Q, i.e. the intersection of $(fX = C)$ and Q contains a limit point of Q.*

COROLLARY. *Every hyperplane of support of a body Q passes through one of its limit points (in particular, each hyperplane of support of a polygon passes through one of its vertices).*

5. Support functions and conjugate spaces

Let the norm $||X|| = \varphi_n(X)$ be introduced into space E_n and let Q be the *closed unit sphere: $Q = (||X|| \leqq 1)$; Q is a convex body in E_n.*

If

$$lX = \sum_{i=1}^{n} l_i X_i$$

is a linear form in $E_n = E_{n\varphi}$, the corresponding hyperplane of support of Q is given by the equation

$$lX = C (= \max_{||X|| \leqq 1} lX). \tag{2.89}$$

Every linear form lX is a vector l with components $l_1, l_2, ..., l_n$. The set of such forms makes up an n-dimensional linear system L_n. A norm can be introduced into L_n, i.e. we can put

$$||l||^* = \psi(l) = \max_{||X|| \leqq 1} lX. \tag{2.90}$$

This norm satisfies, along with the norm $||X||$, conditions (1), (2), (3) (see sec. 3) or (1), (2'), (3).

We have:

$$|lX| \leqq ||l||^* \, ||X||. \tag{2.91}$$

This inequality is a generalization of inequality (2.7) for the Euclidean metric. Equation (2.89) can be written as

$$lX = \|l\|^*. \tag{2.92}$$

Conjugate Spaces. The space $L_n = E_{n\psi}$ thus introduced, of linear forms in $E_{n\varphi}$, is described as *conjugate* to the initial space $E_{n\varphi}$.

The following notation is used:

$$E_{n\psi} = E^*_{n\varphi}. \tag{2.93}$$

The conjugate to a Euclidean space is also Euclidean.

THEOREM 13 (Minkovskii). *If the space $E_{n\psi}$ is regarded as the initial space, its conjugate becomes $E_{n\varphi}$*

$$E^*_{n\psi} = E_{n\varphi} \tag{2.94}$$

(the conjugate to the conjugate space is the initial space).

The reciprocal property whereby spaces $E_{n\varphi}$ and $E_{n\psi}$ are conjugates of each other is known as *reflexivity.* Functions $\psi(X)$ and $\psi(l)$ are said to be *reciprocal;* they are connected by the relationships

$$\max_{\varphi(X)\leqq 1} lX = \psi(l), \qquad \max_{\psi(l)\leqq 1} lX = \varphi(X). \tag{2.95}$$

Similarly, the convex bodies Q and Q^*, defined by the inequalities $\varphi X \leqq 1$, $\psi l \leqq 1$, are also said to be *reciprocal.* (Minkovskii's theorem is no longer in general true for infinite spaces.)

EXAMPLE 16. Let $l_{n,p} (p \geqq 1)$ be an n-dimensional space with the norm

$$\|X\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

When $p > 1$, we have $l^*_{n,p} = l_{n,q}$, where $(1/p)+(1/q) = 1$.

When $p = 2$, (the case of a Euclidean norm) we have $q = 2$ (the n-dimensional Euclidean space $l_{n,2}$ is self-conjugate: $l^*_{n,2} = l_{n,2}$).

EXAMPLE 17. Let $l_{n,1}$ be an n-dimensional space with the norm

$$\|X\| = \sum_{i=1}^{n} |x_i|,$$

and m_n an n-dimensional space with the norm

$$\|Y\| = \max(|y_1|, |y_2|, \ldots, |y_n|);$$

we have:

$$l^*_{n,1} = m_n, \quad m^*_n = l_{n,1}.$$

Every linear function YX in $E_{n\varphi}$ can be regarded as the scalar product of a vector X of $E_{n\varphi}$ and Y of $E_{n\psi}$.

6. Fundamental theorems on support hyperplanes

THEOREM 14. *Let Q be a convex body in n-dimensional space E_n. A hyperplane of support of Q can be drawn through every point of the boundary of Q.*

The following is a generalization of this theorem:

THEOREM 15. *Let L_k $(k < n)$ be a k-dimensional plane drawn through an interior point A of the n-dimensional convex body of E_n. It cuts from Q a k-dimensional convex body Q_k. Let L_{k-1} be the $(k-1)$-dimensional plane in L_k which is a support plane for Q_k. An $(n-1)$-dimensional hyperplane L_{n-1} can be drawn in E_n which is a support hyperplane for Q and contains L_{k-1}.*

Let l_kX be a linear form in the k-dimensional manifold $E_k \subset E_n$ $(1 \leqq k < n)$. The linear form l_nX in the whole space E_n is called an *extension* of the form l_k if, for $X \in E_k$,

$$l_nX = l_kX.$$

The form l_k in L_k has the norm

$$\| l_k \|_{L_k} = \max_{\| X \| \leqq 1,\ X \in E_k} l_kX.$$

The form l_n in E_n has the norm

$$\| l_n \| = \| l_n \|_{L_n} = \max_{\| X \| \leqq 1} l_nX.$$

It follows from this that $\| l_n \| \geqq \| l_k \|_{L_k}$, i.e. when a linear form is extended, its norm can only increase.

THEOREM 16. *Any linear form l_kX, defined in a k-dimensional manifold L_k of E_n, $1 \leqq k < n$, can be extended to the whole of space E_n without changing the norm.*

This theorem can be extended to infinite spaces (the *Hahn–Banach Theorem*).

7. The connection between reciprocal convex bodies

Let Q and Q^* be unit spheres in E_n and E_n^* . A duality (not one-to-one) can be established between the points of their boundaries.

Equation (2.89) of a plane of support of Q in E_n can be written as

$$Y_0X = \sum_{i=1}^{n} y_{0i}x_i = 1.$$

In this case (see (2.92)), $||Y_0||^* = 1$. Formula (2.92) becomes

$$Y_0X = ||Y_0||^* = 1. \tag{2.96}$$

We associate with every point X_0 of E_n, for which $||X_0|| = 1$ (i.e. points of the boundary of Q), all the planes of support of Q passing through it, these planes having equations of the form (2.96). We have:

$$Y_0X_0 = 1, \quad ||X_0|| = ||Y_0||^* = 1. \tag{2.97}$$

Formula (2.97) yields an expression for the $Y_0 \in {}^*{}_n$, which correspond to a given $X_0 \in E_n$; it has symmetry with respect to X_0 and Y_0, which points to the reciprocity of this mapping.

Theorem 14 shows that every X_0 of the boundary of Q ($||X_0||= 1$) has at least one associated Y_0 of the boundary of Q^* ($||Y||^* = 1$).

EXAMPLE 18. On the plane m_2, the sphere

$$(||X|| = \max(|x_1|, |x_2|) \leqq 1)$$

is the square $Q=B_1B_2B_3B_4$; on the plane $l_{1,2}$, the sphere $(||Y||= = |y_1|+|y_2| \leqq 1)$ is the square $Q^*=A_1A_2A_3A_4$. Corresponding to the point X_0 (1, 1) $= B_1$ of the boundary of the square Q, we have the $Y_0(y_1, y_2)$ for which

$$1 = Y_0X_0 = y_1x_1+y_2x_2 = y_1+y_2; \quad ||Y_0|| = |y_1|+|y_2| = 1,$$
$$\text{i.e. } y_1=|y_1| \geqq 0, \ y_2=|y_2| \geqq 0.$$

These points thus fill the side A_1A_2 of the square Q^*.

The side B_1B_2 of the square Q lies on the support line $X_1 = 1$, i.e. $1.X_1+0.X_2 = 1$. The corresponding point is $Y_0(1, 0)$ – the vertex A_2 of the square Q^*. The remaining sides of Q^* correspond to the remaining vertices of Q, the vertices of Q^* to the sides of Q and vice versa.

In three-dimensional space the convex bodies Q and Q^* can only be convex polyhedra simultaneously, the vertices of Q being associated with the faces of Q^*, the ribs of Q with the ribs of Q^*, the faces of Q with the vertices of Q^* and vice versa. Such polyhedra are said to be *reciprocal*. For instance, if Q is a cube, Q^* is an octahedron (and vice versa); if Q is a dodecahedron, Q^* is an icosahedron (and vice versa).

In the n-dimensional case Q and Q^* can only be polyhedra simultaneously, the k-dimensional faces of Q $(0 \leqq k \leqq n-1)$ being associated with the $(n-k-1)$-dimensional faces of Q^*.

8. The cone. The tangent cone

A *cone* K in E_n with vertex at $X_0 \in E_n$ is a set of points of E_n, different from the whole of E_n, and such that, if X belongs to K, the entire ray (tX), $0 \leqq t < \infty$, belongs to K. We shall assume without any proviso that the cone K is a convex body. A cone, together with two of its rays forming an acute angle, contains the entire angle. Cones on a plane are angles not exceeding π. In three-dimensional space examples of cones are provided by dihedral angles not exceeding π, ordinary circular cones or regular pyramids, continued indefinitely, etc.

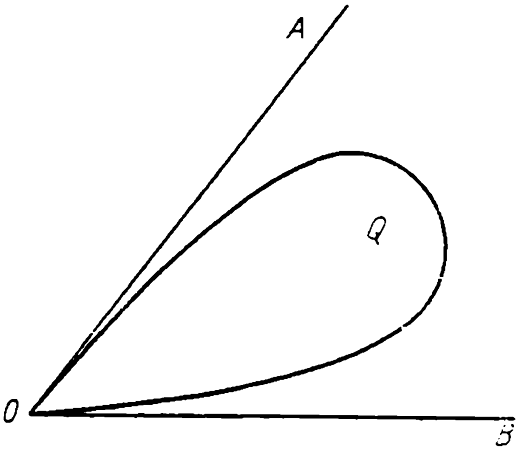

FIG. 3.

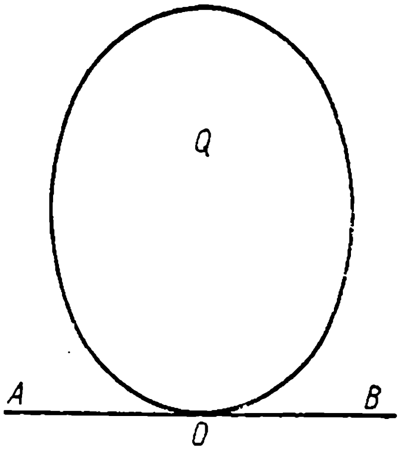

FIG. 4.

Let X_0 be a point of the boundary of a convex body Q in E; let us write $K(X_0)$ for the least cone with vertex at X_0 that contains Q; this cone consists of all the rays joining X_0 with points of Q and all the limiting rays. We shall call the boundary of this cone the *tangent* *cone hypersurface* to Q at the point X_0; two cases are possible:

(1) The cone $K(X_0)$ coincides with the entire half-space, its boundary is the unique hyperplane of support at X_0 to Q, which "*touches*" Q at the point X_0.

(2) The cone $K(X_0)$ is a regular part of the half-space; an infinite set of hyperplanes of support of Q passes through the point X_0. Such a point will be called a *point of sharpening*.

EXAMPLE 19. On a plane, the cone $K(X_0)$ becomes an angle bounding two tangents to Q at the point $X_0 = 0$ (the rays OA, OB in Fig. 3); if this angle is equal to π, both rays form a unique support line (the line $A'B'$ in Fig. 4), tangential to the boundary of Q at the point O. If the angle is less than π, the point O is a point of sharpening, and an infinite set of support lines pass through O.

EXAMPLE 20. Let Q be a convex polyhedron in E_3. If the point X_0 lies inside a face, $K(X_0)$ is the half-space bounded by the plane of the face, which is the unique support plane of Q at X_0. If X_0 lies inside a rib AB, $K(X_0)$ is a dihedral angle less than π, formed by the planes of the faces that intersect in AB; there is an infinite set of support planes to Q at X_0, all of which pass through the rib AB. If X_0 is a vertex of Q, $K(X_0)$ is a polyhedral angle with vertex at X_0, bounded by the planes of the faces that meet in X_0; points of the ribs, and all the more the vertices of Q, are points of sharpening.

9. Helly's theorem

An interesting theorem on the intersection of convex bodies must be mentioned.

THEOREM 17. *Let* $\{Q\}$ *be an arbitrary set of convex bodies given in* E_n, *at least one of which is bounded. If any* $n+1$ *of them have a common point, there is a point common to all the bodies of* $\{Q\}$.

For instance, if an arbitrary set of segments, such that any pair has a common point, is given on a straight line, there is a point common to all the segments.

10. Linear operations on sets

DEFINITION. *Let A and B be any sets of E_n. The vector sum $A+B$ of these sets is defined as the set (X) of points of E_n expressible in the form $X = X_1+X_2$, where $X_1 \in A$, $X_2 \in B$.*

EXAMPLE 21. If A is an arbitrary set, a a point (vector), then $A+a$ is the set obtained by parallel displacement of the set A along the vector a.

EXAMPLE 22. If A is the x-axis, B the y-axis, $A+B$ is the entire plane.

EXAMPLE 23. If A in E_n is an n-dimensional closed sphere of radius ϱ with centre at θ, $A+B$ is a "layer" of thickness ϱ round B, i.e. the set of points of E_n at a distance less than or equal to ϱ from B.

THEOREM 18. *The vector sum of convex sets (bodies) is a convex set (body).*

DEFINITION. *If λ is a number, A a set in E_n, λA is the set of all points of the form λX, where $X \in A$.*

A *similitude transformation* of the set A with transformation coefficient $\lambda > 0$ is a transformation of the set A into the set λA. A *symmetric mapping* of A with respect to the centre ϑ is a transformation of A into $-A$.

When $\lambda = 0$ the set λA consists of the single point θ.

If A is a convex set (body), λA is a convex set (body).

If A_i $(i = 1, 2, \ldots, n)$ are convex sets (bodies), $\sum_{i=1}^{n} \lambda_i A_i$ is a convex set (body) for any system of non-negative numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let T_1 $(l = C_1)$ and T_2 $(l = C_2)$ be two parallel hyperplanes of support of the convex bodies Q_1 and Q_2, while $(l_1 \leqq C_1)$, $(l_1 \leqq C_2)$ are the corresponding half-spaces containing Q_1 and Q_2 respectively; t_1, t_2 are arbitrary positive numbers. Now, $t_1T_1+t_2T_2$ is a hyperplane of support parallel to them $(l = t_1C_1+t_2C_2)$ of the body $Q = t_1Q_1+t_2Q_2$, lying in the half-space $(l \leqq t_1C_1+t_2C_2)$.

THEOREM 19 (Brunno–Minkovskii). *Let A_0 and A_1 be convex bodies in E_n, and $A_t = tA_0+(1-t)A_1$, $0 \leqq t \leqq 1$, a "linear system" of convex bodies; the n-dimensional volumes J_n of the bodies A_0, A_1 and all the A_t are connected by the Brunno–Minkovskii inequality*

$$\sqrt[n]{J_n(A_t)} \geqq (1-t)\sqrt[n]{J_n(A_0)} + t\sqrt[n]{J_n(A_1)}. \qquad (2.98)$$

The case of equality occurs if and only if A_0 and A_1 are homothetic convex bodies, i.e. one is obtained from the other by a similitude transformation and parallel displacement ($A_1 = \tau A_0 + b$, $\tau \geqq 0$, b is a vector).

Inequality (2.98) was proved by Brunno (in 1887); the case of equality was proved by G. Minkovskii (in 1891).

Inequality (2.98) implies that $\sqrt[n]{J_n(A_t)}$ is a concave function for any t in $[0, 1]$.

The theorem still holds for non-convex bodies, and in general for any sets, if $J_n(A)$ is understood as the exterior n-dimensional measure of the set A or the measure when it is measurable (see ref. 7).

The Brunno–Minkovskii theorem is employed in proving the isoperimetric and numerous other geometric properties of convex bodies.

CHAPTER III

SERIES

Introduction

Infinite sequences and their limits have been discussed in Chapter I. *Infinite series*, or simply *series*, are closely connected with sequences, e.g.

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n \tag{3.1}$$

is "the sum of an infinite number of terms".

Series are commonly employed in the most widely spread branches of mathematical analysis and in the solution of applied problems, since they represent one of the most universal and effective means both of investigation and computation.

Examples of series (infinite geometrical progressions) were known to the mathematicians of antiquity. In the process of developing the analysis of infinitesimals, series made their appearance, with the aid of which the values of various functions could be computed: Mercator's series for the logarithm, Newton's series for sin x, cos x, arc sin x, arc cos x, $(1+x)^{\alpha}$, etc. Series were widely used by Euler for representing functions in various works; in addition to power series, trigonometric series were employed by Euler. He also made use of divergent series, and seems to have been the first to be concerned with improving the convergence of series. The vast amount of factual material on series that had accumulated by the beginning of the nineteenth century posed to scientists the problem of finding a strict basis for the theory of series. The investigations of Abel, Gauss, Cauchy and others in this direction played an important part in laying the foundations of mathematical analysis as a whole.

The present chapter is devoted to the basic theory and practice of evaluating series. Numerical series are considered in § 1, functional in § 2, and the various methods of computing them in § 3. (See Chapter IV for vector series.)

1. Basic concepts

DEFINITION: *A series is defined as in* (3.1), *composed of the terms of an infinite sequence* $\{a_n\}$ $(n = 1, 2, 3, \dots)$. For every series there is a corresponding sequence $\{s_n\}$ $(n = 1, 2, 3, \dots)$ of *partial sums*, where

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad \dots, \quad s_n = \sum_{k=1}^{n} a_k, \quad \dots . \tag{3.2}$$

EXAMPLE 1. Given the series

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots = \sum_{n=1}^{\infty} a_k, \dots$$

we have

$$s_1 = 1, \quad s_2 = \frac{4}{3}, \quad \dots, \quad s_n = \frac{3}{2}\left(1 - \frac{1}{3^n}\right), \dots .$$

The following two cases are possible when considering the sequence of partial sums.

Case 1. The sequence $\{s_n\}$ has a definite finite limit

$$S = \lim_{n \to \infty} s_n.$$

This limit is called the *sum of series* (3.1) and is written as

$$S = \sum_{n=1}^{\infty} a_n. \tag{3.3}$$

In Example 1,

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} = \frac{3}{2}.$$

Case 2. The sequence $\{s_n\}$ has no finite limit.

EXAMPLE 2.

$$\left.\begin{aligned} &1 + 2 + 3 + \dots + n + \dots = \sum_{n=1}^{\infty} n; \\ &s_1 = 1, \quad s_2 = 3, \quad \dots, \quad s_n = \frac{(1+n)n}{2}, \dots \end{aligned}\right\}$$

As $n \to \infty$, the partial sum $s_n \to \infty$.

EXAMPLE 3.

$$1-1+1-1+\ldots = \sum_{n=1}^{\infty}(-1)^{n-1}$$

$$s_1 = 1,\ s_2 = 0,\ s_3 = 1,\ \ldots,\ s_n = \begin{cases} 0 & \text{when } n \text{ is even} \\ 1 & \text{when } n \text{ is odd.} \end{cases}$$

EXAMPLE 4.

$$1-2+4-8+\ldots = \sum_{n=1}^{\infty}(-2)^{n-1};$$

$$s_1 = 1,\quad s_2 = -1,\quad s_3 = 3,\quad s_4 = -5,\ \ldots,\quad s_n = \frac{1-(-2)^n}{3}.$$

In Case 1 the series is said to be *convergent*, whereas it is *divergent* in Case 2 (Examples 2, 3, 4).

Example 2 illustrates an important particular case of a divergent series, when $s_n \to \infty$ (or $s_n \to -\infty$); Example 3 illustrates the case when s_n *oscillates* while remaining bounded, and Example 4 the case when s_n *oscillates* but is unbounded.

We shall only discuss convergent series in this section. In the case of a convergent series (3.1), the sum S can be written as

$$S = s_n + R_n, \tag{3.4}$$

where

$$R_n = a_{n+1} + a_{n+2} + \ldots = \sum_{k=n+1}^{\infty} a_k \tag{3.5}$$

is called the *remainder* or *remainder term* of the series.

Evaluation (or *summation*) of a series implies finding its sum S. We must therefore verify the convergence of a series before finding its sum.

Strict summation (some methods of which will be given below) is only possible for a narrow class of series. For the majority, the sum is found approximately. In approximate summations, the sum S is replaced by the partial sum s_n.

It follows from the definition of the sum of a convergent series that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}(S-s_n) = 0. \tag{3.6}$$

This indicates that, given a sufficiently large n, the absolute value of the remainder term R_n can be made as small as desired.

To ensure the required accuracy when replacing S by s_n, it becomes necessary to estimate the difference

$$S - s_n = R_n,$$

i.e. the remainder term.

In addition, practical convenience in computations requires that the number n of terms of the partial sum s_n, required for achieving the necessary accuracy, be not too large. Reasonably accurate summation of certain series would require taking thousands, or even tens of thousands, of terms. Such series are said to be *slowly convergent.*

EXAMPLE 5. To compute the sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

to an accuracy of 0·001, the summation of a thousand terms is needed.

The question thus arises of improving the convergence of a series, i.e. transforming it in such a way that summation of the transformed series with the same accuracy requires a smaller number of terms in the partial sum s_n. The fundamental questions for the summation of series are therefore: convergence, estimation of the remainder term, improvement of the convergence.

2. Some convergence tests for series

Cauchy's test for the convergence of a sequence (3.2) takes the following form as applied to series.

CAUCHY'S TEST. *The necessary and sufficient condition for convergence of the series* (3.1) *is that, given any* $\varepsilon > 0$, *there exists an* n_ε *such that*

$$|a_{n+1} + a_{n+2} + \ldots + a_{n+m}| < \varepsilon$$

for all $n > n_\varepsilon$ *and any integer* m.

An important corollary follows from this test with $m = 1$.

If series (3.1) *is convergent, its general term* a_n *tends to zero as* n *increases:*

$$a_n \to 0 \quad \text{as} \quad n \to \infty.$$

This condition is necessary but not sufficient for convergence.

EXAMPLE 6. The *harmonic series*

$$1+\frac{1}{2}+\frac{1}{3}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n}$$

is divergent, although $a_n = 1/n \to 0$ as $n \to \infty$.

Cauchy's test is the most general necessary and sufficient criterion for the convergence or divergence of a series.

Direct application of it is difficult, however. The theory of series includes a whole range of sufficient tests for convergence and divergence of varying degrees of generality, and widely employed. Historically, the first of these is

D'ALEMBERT'S TEST. *If the limit*

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=q$$

exists for series (3.1), *the series is convergent when* $q<1$ *and divergent* $q>1$. *(The test is inconclusive when* $q=1$*).*

A more general test is

THE CAUCHY–HADAMARD TEST. *We write*

$$q=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}.$$

Series (3.1) *is now convergent when* $q<1$ *and divergent when* $q>1$. *(The test is inconclusive when* $q=1$.*)*

THE PRINCIPLE OF MAJORANT SERIES. The series

$$b_1+b_2+\ldots=\sum_{n=1}^{\infty}b_n \tag{3.7}$$

with non-negative terms b_n is said to be *majorant* for series (3.1) if, given any n,

$$|a_n|\leqq b_n$$

as from some $n \geqq N$.

The convergence of series (3.7) *implies the convergence of series* (3.1).

Numerous convergence tests are based on the principle of majorant series, including in particular the d'Alembert and Cauchy–Hadamard tests.

§ 1. Numerical series

1. Alternating series and series of constant sign

If all the terms of a series are of the same sign, the series is said to be *of constant sign.* Such series include those, all the terms of which are positive (positive series), and those, all the terms of which are negative (negative series).

If all the terms of a series are not of the same sign, the series is described as *alternating.* When there is a finite number of terms of the same sign, they can be neglected when investigating the convergence and attention paid only to the remaining series of constant sign. The theory of series with an infinite number of positive and negative terms has certain differences in principle from the theory of series of constant sign.

The following is a particular case of the principle of majorant series.

THEOREM 1. *The series*

$$a_1+a_2+\ldots+a_n+\ldots=\sum_{n=1}^{\infty}a_n \tag{3.8}$$

with terms of arbitrary sign is convergent if the series

$$|a_1|+|a_2|+\ldots+|a_n|+\ldots=\sum_{n=1}^{\infty}|a_n| \tag{3.8*}$$

formed from the absolute values of the terms of series (3.8), *is convergent.*

In this case series (3.8) is described as *absolutely convergent.* All series of constant sign are *absolutely convergent.*

Cases are possible when series (3.8) is convergent, and series (3.8*) divergent. Series (3.8) is then described as *non-absolutely,* or *conditionally* convergent.

EXAMPLE 7. The series

$$1-\frac{1}{2}+\frac{1}{3}-\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

is convergent and its sum is $S=\ln 2$.

On the other hand, the series of the absolute values of the terms is the divergent harmonic series.

2. Properties of convergent series. The associative property

If the terms of the convergent series

$$a_1+a_2+\ldots+a_n+\ldots=\sum_{n=1}^{\infty} a_n$$

are grouped in any manner, without changing their order:

$$a_1+\ldots+a_{n_1},\ a_{n_1+1}+\ldots+a_{n_2},\ a_{n_2+1}+\ldots+a_{n_3},\ \ldots$$
$$\ldots,\ a_{n_{k-1}+1}+\ldots+a_{n_k},\ \ldots,$$

where $\{n_k\}$ is a partial increasing sequence of numbers of the natural series, the series of the sums of the terms of these groups

$$(a_1+\ldots+a_{n_1})+(a_{n_1+1}+\ldots+a_{n_2})+\ldots$$

is always convergent and has the same sum as the initial series. This property can be utilized for improving the convergence of series.

The rearrangement property of absolutely convergent series. If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, the series obtained from it by *any rearrangement* (*commutation*) of the terms is also convergent and has the same sum as the initial series.

Non-absolutely convergent series do not have the commutative property. We have, furthermore,

THEOREM 2 (Riemann). *The terms of a non-absolutely convergent series can be rearranged in such a way that the transformed series has a sum equal to any previously assigned number, or becomes divergent.*

3. General tests for the convergence of series of positive terms

If we want to determine the convergence of the series with positive terms

$$\sum_{k=1}^{\infty} a_k, \tag{3.9}$$

we choose at discretion another series with positive terms

$$\sum_{k=1}^{\infty} b_k \tag{3.10}$$

such that its convergence and sum are already known to us.

We introduce the notation:

$$R_m = \sum_{k=m}^{\infty} a_k; \tag{3.11}$$

R_m is the *remainder term* of the series, which is equal to a finite positive number if series (3.9) is convergent, and to ∞, if series (3.9) is divergent;

$$A_k(l) = \frac{b_k}{a_{k+l}}, \tag{3.12}$$

where l is a parameter taking integral values such that $m+l \geqq 0$, m is a fixed positive integer;

$$\underline{\lim}_{k\to\infty} A_k(l) = \underline{A}(l), \tag{3.13}$$

$$\overline{\lim}_{k\to\infty} A_k(l) = \overline{A}(l). \tag{3.14}$$

Let

$$\underline{\lim}_{n\to\infty} \sum_{k=m}^{n} b_k = \underline{B}_m, \tag{3.15}$$

$$\overline{\lim}_{n\to\infty} \sum_{k=m}^{n} b_k = \overline{B}_m; \tag{3.16}$$

in the case when the lower and upper limits coincide, i.e. when the limit exists, we introduce the notation:

$$\lim_{k\to\infty} A_k(l) = A(l), \tag{3.17}$$

$$\lim_{n\to\infty} \sum_{k=m}^{n} b_k = B_m. \tag{3.18}$$

We have the following sufficient tests for convergence of series (3.9).

Test I. *If* $\underline{A}(l) > 0$ *and* $\underline{B}_m < +\infty$, *series* (3.9) *is convergent, whereas if* $-\infty < \underline{A}(l) \leqq 0$ *and* $\underline{B}_m = -\infty$, *the series is divergent.*

Test II. *If* $\overline{A}(l) < 0$ *and* $\overline{B}_m > -\infty$, *the series is convergent, whereas if* $0 \leqq \overline{A}(l) < +\infty$ *and* $\overline{B}_m = +\infty$, *the series is divergent.*

Test III. *If* $A(l) > 0$ *and* $B_m < +\infty$ *or* $A(l) < 0$ *and* $B_m > -\infty$, *the series is convergent; if* $0 \leqq A(l) < +\infty$ *and* $B_m = +\infty$ *or* $-\infty < A(l) \leqq 0$ *and* $B_m = -\infty$, *the series is divergent.*

By a suitable choice of auxiliary series (3.10), we can obtain as particular cases from Tests I, II and III either new or already familiar sufficient tests for the convergence of tests with positive terms (see sec. 5). These tests include, in particular, the familiar necessary condition for convergence which says that, if series (3.9) is convergent, $\lim\limits_{n\to\infty} a_n = 0$.

4. Remainder term estimates corresponding to the various convergence tests

When one of the convergence Tests I, II, III is satisfied for series (3.9), a corresponding estimate of the remainder term can be given.

(1) For Test I:

$$\frac{B_m}{\inf\limits_{k\geqq m} A_k(l)} \leqq R_{m+l} \leqq \frac{B_m}{\sup\limits_{k\geqq m} A_k(l)}, \tag{3.19}$$

where m is such that all the $A_k(l) > 0$; l is an integral parameter, satisfying the condition $m+l \geqq 0$.

(2) For Test II:

$$\frac{B_m}{\inf\limits_{k\geqq m} A_k(l)} \leqq R_{m+l} \leqq \frac{B_m}{\sup\limits_{k\geqq m} A_k(l)}, \tag{3.20}$$

where m is such that all the $A_k(l) < 0$ and $m+l \geqq 0$.

(3) For Test III, if $A(l) > 0$, inequalities (3.19) hold, while (3.20) hold if $A(l) < 0$.

For Test III, it becomes convenient to utilize the following particular cases of (3.19) and (3.20):

(a) When $A(l) > 0$ and the sequence $\{A_k(l)\}$ is monotonically increasing for $k \geqq m$, we have

$$\frac{B_m}{A(l)} \leqq R_{m+l} \leqq \frac{B_m}{A_m(l)}. \tag{3.21}$$

(b) When $A(l) < 0$ and the sequence $\{A_k(l)\}$ is monotonically decreasing for $k \geqq m$, we have

$$\frac{B_m}{A_m(l)} \leqq R_{m+l} \leqq \frac{BA}{A(l)}. \tag{3.22}$$

(c) When $A(l) < 0$ and the sequence $\{A_k(l)\}$ is monotonically increasing for $k \geqq m$, (3.21) holds.

(d) When $A(l) > 0$ and sequence $\{A_k(l)\}$ is monotonically decreasing for $k \geqq m$, (3.22) holds.

Example 8. Let us estimate the remainder term of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

On putting, say, $b_n = z_{n-1} - z_n$, where $z_n = -1/2n^2$, we have, provided that $m > 1$ and $l \geqq 0$:

$$A_m(l) = \frac{(2m+1)(m+l)^3}{2m^2(m+1)^2},$$

$$A(l) = \lim_{n\to\infty} \frac{b_m}{a_{m+l}} = 1, \quad B_m = \lim_{n\to\infty} (z_{n+1} - z_m) = \frac{1}{2m^2}.$$

It is easily verified that the sequence $\{A_m(0)\}$ $(m = 1, 2, 3, \ldots)$ is monotonically increasing, while the sequence $\{A_m(1)\}$ $(m = 1, 2, 3, \ldots.)$ is monotonically decreasing.

Consequently, when $l = 0$ and $m \geqq 1$, we have by (3.21):

$$\frac{1}{2m^2} < R_m < \frac{(m+1)^2}{(2m+1)m^3},$$

while with $l = 1$, by (3.22),

$$\frac{1}{(2m+1)(m+1)} < R_{m+1} < \frac{1}{2m^2}.$$

We obtain from the last two inequalities, in choosing the greatest lower and least upper values,

$$\frac{1}{(2m-1)m} < R_m < \frac{(m+1)^2}{(2m+1)m^3}.$$

This example shows that, given the same choice of sequence $\{b_n\}$ and different values of the index l, estimates can be obtained that do not overlap each other.

5. Special tests for the convergence of series of positive terms. Estimates of the remainder term

Numerous familiar classical tests for the convergence of series with positive terms follow from the general convergence Tests I, II and III (sec. 3). We shall give here certain of these and the corresponding remainder term estimates (see ref. 9).

1°. THE GENERALIZED D'ALEMBERT TEST. The following two tests can be regarded as a generalization of the d'Alembert test mentioned above.

Test A. *If, for a fixed* l,

$$\overline{A}(l) = \overline{\lim_{n\to\infty}} \frac{a_{n+1}-a_n}{a_{n+l}} < 0, \tag{3.23}$$

the series $\sum_{n=1}^{\infty} a_n$ *is convergent, whereas the series is divergent if*

$$\underline{A}(l) = \underline{\lim}_{n\to\infty} \frac{a_{n+1}-a_n}{a_{n+l}} > 0, \tag{3.24}$$

Test B. *If, for a fixed* l,

$$A(l) = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{a_{n+l}}$$

is negative, the series is convergent; whereas if the limit is positive, the series is divergent.

In particular, when $l = 0$ Tests A and B yield d'Alembert's test (Introduction, sec. 2).

If the convergence of the series $\sum_{n=1}^{\infty} a_n$ is established with the aid of Test A or B, we have on the basis of (3.20):

$$\frac{-a_m}{\inf\limits_{k\geqq m} \frac{a_{k+1}-a_k}{a_{k+l}}} \leqq R_{m+l} \leqq \frac{-a_m}{\sup\limits_{k\geqq m} \frac{a_{k+1}-a_k}{a_{k+l}}}. \tag{3.25}$$

If the sequence $\{A_k(l) = (a_{k+1}-a_k)/a_{k+1}\}$ is monotonically increasing for $k \geqq m$, we have

$$\frac{-a_m}{\frac{a_{m+1}-a_m}{a_{m+l}}} \leqq R_{m+l} \leqq \frac{-a_m}{\lim\limits_{k\to\infty} \frac{a_{k+1}-a_k}{a_{k+l}}}. \tag{3.26}$$

If the sequence $\{A_k(l)\}$ is monotonically decreasing for $k \geqq m$, we have

$$\frac{-a_m}{\lim\limits_{k\to\infty} \dfrac{a_{k+1}-a_k}{a_{k+l}}} \leqq R_{m+l} \leqq \frac{-a_m}{\dfrac{a_{m+1}-a_m}{a_{m+l}}}\,. \tag{3.27}$$

EXAMPLE 9.

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2}\,. \tag{3.28}$$

It is easily verified that, if $l+1 < 0$, the sequence $\{A_k(l) = (a_{k+1} - -a_k)/a_{k+1}\}$ for series (3.28) is negative and monotonically decreasing as from a certain $k \geqq N(l)$. If $l+1 > 0$, the sequence $\{A_k(l)\}$ is negative and monotonically increasing.

We therefore have, on the basis of (3.27) with $l+1 \leqq 0$, as from a certain $m \geqq N(l)$ (say as from $m \geqq 1$ when $l = -1$):

$$\frac{1}{2^{m+l-1}m^2} \leqq R_{m+l} \leqq \frac{(m+l)^2}{2^{m+l-1}(m+l)^2\,(m^2+4m+2)}\,.$$

When $l+1 > 0$ we have to reverse the inequality signs in the above.

For instance, on putting $m+l = 6$, and accordingly assigning the following values to m and l:

m	9	8	7	6	5	4
l	-3	-2	-1	0	1	2

it may be seen that the best upper estimate is obtained with $l = -1$, and the best lower estimate with $l = 0$, so that in this case $0{\cdot}000686 \leqq \leqq R_6 \leqq 0{\cdot}000703$.

2°. THE GENERALIZED CAUCHY TEST. On putting $b_n = \varrho^n$ and applying Tests, I, II (sec. 3), we get the following convergence test, which may be regarded as a generalization of the Cauchy–Hadamard test.

Test C. *If, for a fixed* l *and* $\varrho < 1$,

$$\varliminf_{n\to\infty} \frac{\varrho^n}{a_{n+l}} > 0,$$

the series $\sum_{n=1}^{\infty} a_n$ *is convergent; whereas the series is divergent if, for a given* l *and* $\varrho > 1$,

$$0 \leqq \varlimsup_{n\to\infty} \frac{\varrho^n}{a_{n+l}} < \infty.$$

When $l = 0$, this test yields the familiar Cauchy–Hadamard test (Introduction, sec. 2).

If the convergence of the series can be established with the aid of Test C, we have by (3.19), for any $m+l > 0$,

$$\frac{\varrho^m}{(1-\varrho)\sup\limits_{k\geqq m}\dfrac{\varrho^k}{a_{+l}}} \leqq R_{m+l} \leqq \frac{\varrho^m}{(1-\varrho)\inf\limits_{k\geqq m}\dfrac{\varrho^k}{a_{k+l}}}. \tag{3.29}$$

Here, $\varrho < 1$ is an arbitrary number, for which $\varliminf\limits_{n\to\infty} (\varrho^n/a_{k+1}) > 0$. If the sequence $\{A_k(l) = \varrho^k/a_{k+1}\}$ is monotonically increasing for $k \geqq m$, we have

$$\frac{\varrho^m}{(1-\varrho)\lim\limits_{k\to\infty}\dfrac{\varrho^k}{a_{k+l}}} \leqq R_{m+l} \leqq \frac{a_{m+l}}{1-\varrho}. \tag{3.30}$$

Whereas, if the sequence $\{A_k(l)\}$ is monotonically decreasing for $k \geqq m$, we have

$$\frac{a_{m+l}}{1-\varrho} \leqq R_{m+l} \leqq \frac{\varrho^m}{(1-\varrho)\lim\limits_{k\to\infty}\dfrac{\varrho^k}{a_{k+l}}}. \tag{3.31}$$

EXAMPLE 10.

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n}} \qquad (0 \leqq x < 1). \tag{3.32}$$

It is easily shown that estimate (3.31) can be applied to series (3.32) on condition that $m+l \geqq 3$.

On taking $\varrho = x$, we have

$$\frac{x^{m+l}}{(1-x)\sqrt[m+l]{m+l}} \leqq R_{m+l} \leqq \frac{x^{m+}}{1-x}.$$

3°. RAABE'S TEST. A particular case of the general Test III (sec. 3) is

Test D. *The series* $\sum_{n=1}^{\infty} a_n$ *is convergent if, for any integral* $l \geqq 0$,

$$A(l) = \lim_{n\to\infty} \frac{na_{n+1}-(n-1)a_n}{a_{n+l}} < 0, \tag{3.33}$$

and is divergent if

$$A(l) > 0.$$

When $l = 0$, we get

Raabe's Test. *The series* $\sum_{n=1}^{\infty} a_n$ *is convergent if*

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) > 1,$$

and is divergent if

$$n\left(1-\frac{a_{n+1}}{a_n}\right) \leqq 1.$$

for all $n \geqq n_0$ *say.*

Example 11.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}. \tag{3.34}$$

The convergence of this series is readily established by applying Test D. We can also make use here of the estimates for the remainder term of sec. 4, corresponding to cases (3), (c) and (3), (d).

On applying estimate (3), (c) (sec. 4) with $m \geqq 9$, and $1 \geqq 2$, we get

$$\frac{(m-1)(m+3)}{(2m-3)(m+l)(m+l+1)(m+l+2)} \leqq R_{m+l} \leqq \frac{m-1}{(m+1)(m+2)}. \tag{3.35}$$

On applying estimate (3), (d) (sec. 4) with $m+l > 0$, $l \leqq 1$ and $m > 4l/(2l-3)$, we have

$$\frac{(m-1)(m+3)}{(2m-3)(m+l)(m+l+1)(m+l+2)} \geqq R_{m+l} \geqq \frac{m-1}{2m(m+1)(m+2)}. \tag{3.36}$$

On comparing inequalities (3.35) and (3.36), say with $m+l = 12$, it is easily seen that

$$0{\cdot}00315 \leqq R_{12} \leqq 0{\cdot}00337,$$

he true value here being $R_{12} = 0{\cdot}00320$.

4°. Gauss's Test. If $\lim_{n\to\infty} a_{n+1}/a_n = 1$, d'Alembert's test gives no answer as regards the convergence of the series, as mentioned above. If the ratio a_{n+1}/a_n has the form

$$\frac{a_{n+1}}{a_n} = \frac{n^\lambda + pn^{\lambda-1} + \varphi(n)}{n^\lambda + qn^{\lambda-1} + \psi(n)},$$

where $\varphi(n)$ and $\psi(n)$ have lower orders than $n^{\lambda-1}$, the convergence or divergence of the series may be established with the aid of the following test.

Gauss's Test. *The series* $\sum_{n=1}^{\infty} a_n$ *is convergent when* $q-p > 1$, *and divergent when* $q-p \leqq 1$.

Example 12. When $x = 1$, the *hypergeometric series* (ref. 7):

$$\frac{\alpha\beta}{1.\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)} + \dots \quad (3.37)$$

is convergent provided that

$$\gamma > \alpha+\beta .$$

Notice that, from a certain n onwards, series (3.37) is of constant sign for any real α, β, γ (it is assumed that none of the numbers α, β, γ is negative).

We have in the present case, say with $l = 0$,

$$A_n(0) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma+n} - (\gamma-\alpha-\beta)$$

and

$$A(0) = -(\gamma-\alpha-\beta) < 0.$$

On observing that, for all $n > -\gamma$, the sequence $\{A_n(0)\}$ will be monotonically increasing or monotonically decreasing, and making use of (3.26) and (3.27), we get

$$\frac{(m+\gamma)ma_m}{(m+\gamma)(\gamma-\alpha-\beta)-(\gamma-\alpha)(\gamma-\beta)} \overset{\leqq}{\geqq} R_m \overset{\leqq}{\geqq} \frac{ma_m}{\gamma-\alpha-\beta},$$

$$(m > -\gamma),$$

where

$$a_m = \frac{\alpha(\alpha+1)\dots(\alpha+m-1)\beta(\beta+1)\dots(\beta+m-1)}{m!\,\gamma(\gamma+1)\dots(\gamma+m-1)}.$$

5°. CAUCHY'S INTEGRAL TEST. Let the general term of the series $\sum\limits_{n=1}^{\infty} a_n$ be monotonically decreasing with increasing n. Obviously, it is now possible to choose an infinite set of positive functions $a(x)$, monotonically decreasing and continuous for $x > 0$, and such that

$$a(n) = a_n.$$

Let $a(x)$ be one of these functions. The series $\sum\limits_{n=1}^{\infty} a_n$ is now convergent or divergent, according as the improper integral

$$\int_{\alpha}^{\infty} a(x)\,dx, \tag{3.38}$$

where $\alpha \geqq 1$, is convergent or divergent.

This test follows as a particular case from general Tests I and II (sec. 3), if we put say

$$b_n = \int_{n}^{n+1} a(x)\,dx \quad \text{and} \quad l = 0.$$

EXAMPLE 13. We are given the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^{1+\sigma} n} \qquad (\sigma > 0). \tag{3.39}$$

We choose as $a(x)$ the function $1/(x \ln^{1+\sigma} x)$ and put $\alpha = 2$, so that

$$\int_{2}^{\infty} \frac{1}{x \ln^{1+\sigma} x}\,dx = -\frac{1}{\sigma \ln^{\sigma} x}\bigg|_{2}^{\infty} = \frac{1}{\sigma \ln^{\sigma} 2};$$

the series is therefore convergent.

EXAMPLE 14. We are given the series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln \ln n}. \tag{3.40}$$

We choose $a(x) = 1/(x \ln x \ln \ln x)$ and put $\alpha = 3$:

$$\int_{3}^{\infty} \frac{1}{x \ln x \ln \ln x}\,dx = \ln \ln \ln x\bigg|_{3}^{\infty} = \infty,$$

so that the series is divergent.

EXAMPLE 15. Given the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}. \tag{3.41}$$

we choose $a(x) = 1/x^{\sigma}$ and put $\alpha = 1$. On recalling that

$$\int \frac{dx}{x^{\sigma}} = \begin{cases} \dfrac{x^{1-\sigma}}{1-\sigma} & \text{for} \quad \sigma \neq 1, \\ \ln x & \text{for} \quad \sigma = 1, \end{cases}$$

we get:

$$\int_{1}^{\infty} \frac{dx}{x^{\sigma}} = \begin{cases} \dfrac{1}{\sigma-1} & \text{for} \quad \sigma > 1, \\ \infty & \text{for} \quad \sigma \leqq 1. \end{cases}$$

The series is thus convergent for $\sigma > 1$ and divergent for $\sigma \leqq 1$. Notice the following points:

(1) In view of the fact that the choice of the function $a(x)$ is to a large extent arbitrary, an infinite set of estimates can in general be quoted, corresponding to the test.

(2) When the sequence $\{a_n\}$ is monotonic, estimates corresponding to Cauchy's integral test can be used, independently of the particular test used to prove the convergence of the series $\sum_{n=1}^{\infty} a_n$.

EXAMPLE 16.

$$\sum_{n=1} \frac{n}{a^n} \qquad (a > 1).$$

Let $a(x) = x/a^x$, say. The function $a(x)$ is monotonically decreasing for $x > 1/\log a$. Therefore, on evaluating

$$A_k = \frac{a^k}{k} \int_{k}^{k+1} x a^{-x}\, dx = \frac{a-1}{a \ln a} + \frac{a - \ln a - 1}{k a \ln^2 a}$$

and making use of estimate (3.26), we have

$$\frac{m(m \ln a + 1)}{a^{m-1}(a-1)\left(m \ln a + 1 - \dfrac{\ln a}{a-1}\right)} \leqq R_m \leqq \frac{m \ln a + 1}{a^{m-1}(a-1) \ln a},$$
$$(m \geqq 1).$$

6°. N. V. **Bugaev's Theorem**; V. P. **Ermakov's Test.** Let $a(x)$ be the function introduced in the statement of Cauchy's integral test, and $\delta(x)$ some positive differentiable function, increasing as x increases and such that $\lim_{x \to \infty} 1/\delta(x) = 0$. The following theorem can now be obtained on the basis of Cauchy's integral test.

Theorem 3 (N. V. Bugaev). *If the function* $\delta'(x)\, a[\delta(x)]$ *is monotonic for sufficiently large* x, *the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} \delta'(n) a[\delta(n)]$ *are simultaneously convergent or divergent.*

Thus any convergent test applied to the series $\sum_{n=1}^{\infty} \delta'(n) a[\delta(n)]$ will at the same time yield a convergence test for the series $\sum_{n=1}^{\infty} a_n$.

In particular, we have the following corollary of N. V. Bugaev's Theorem.

V. P. **Ermakov's Test.** *If*

$$\lim_{m \to \infty} \frac{e^m a(e^m)}{a(m)} < 1,$$

the series $\sum_{n=1}^{\infty} a_n$ *is convergent; whereas if*

$$\lim_{m \to \infty} \frac{e^m a(e^m)}{a(m)} > 1,$$

the series is divergent.

On putting say $b_n = \int_n^{n+1} \delta'(x) a[\delta(x)]\, dx$ and using the general tests of sec. 4, an infinite set of estimates can be obtained (for the remainder term of the series $\sum_{n=1}^{\infty} a_n$), corresponding to different methods of choosing the functions $\delta(x)$.

7°. **Lobachevskii's Test.** *If the terms of the series*

$$\sum_{n=1}^{\infty} a(n)$$

are monotonically decreasing, then the series is convergent or divergent at the same time as the series

$$\sum_{m=1}^{\infty} p_m 2^{-m},$$

where p_m is given by

$$a(p_m) \leqq 2^{-m}, \quad a(p_m+1) \leqq 2^{-m}.$$

We can also define p_m from the equation

$$a(p_m) = 2^{-m},$$

if the function $a(x)$ is monotonic and defined for any value of x.

EXAMPLE 17.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The equation

$$1/p_m^2 = 2^{-m}$$

gives us $p_m = 2^{\frac{1}{2}m}$; we form the series

$$\sum_{m=1}^{\infty} p_m 2^{-m} = \sum_{m=1}^{\infty} 2^{-\frac{m}{2}}.$$

This is a convergent geometrical progression, so that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

s also convergent.

6. The convergence of alternating series

Given the alternating series (3.8), we form series (3.8*) from the absolute values of its terms.

To establish the absolute convergence of series (3.8), we can apply to the positive series (3.8*) all the tests described above for series of constant sign. As already mentioned, however, tests for the convergence of series (3.8*) may not be valid for series (3.8).

Alternating numerical series. Alternating series are those, the terms of which are alternately positive and negative. The following holds for such series:

THEOREM 4 (Leibniz). *If the terms of an alternating series $\sum_{n=1}^{\infty} a_n$ are monotonically decreasing in absolute value,*

$$|a_{n+1}| \leqq |a_n| \qquad (n = 1, 2, 3, \ldots)$$

and the n-th term tends to zero,

$$\lim_{n\to\infty} a_n = 0,$$

the series is convergent.

Estimation of the remainder term of an alternating series is simple and effective.

The remainder term of an alternating series that satisfies the conditions of Leibniz's theorem,

$$R_n = a_{n+1}+a_{n+2}+\ \ldots,$$

has the sign of its first term a_{n+1} and is less than it in absolute value:

$$|R_n| = a_{n+1}+a_{n+2}+\ \ldots\ .$$

The criteria for convergence of Abel and Dirichlet are more general than Leibniz's test.

Suppose we have the series

$$\sum_{n=1}^{\infty} a_n b_n = a_1b_1+a_2b_2+\ \ldots\ +a_nb_n+\ \ldots, \tag{3.42}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers.

ABEL'S TEST. *Series* (3.42) *is convergent if the series*

$$\sum_{n=1}^{\infty} b_n = b_1+b_2+\ \ldots\ +b_n+\ \ldots \tag{3.43}$$

is convergent, while the numbers a_n form a monotonic and bounded sequence:

$$|a_n| < K \qquad (n = 1, 2, 3, \ldots).$$

EXAMPLE 18.

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)\frac{1}{3^{n-1}}; \qquad a_n = 1-\frac{1}{2^n} < 1, \qquad b_n = \frac{1}{3^{n-1}}.$$

DIRICHLET'S TEST. *Series* (3.42) *is convergent if the partial sums of series* (3.43) *are bounded in aggregate:*

$$|s_n| \leqq M \qquad (n = 1, 2, 3, \ldots), \tag{3.44}$$

while the numbers a_n form a monotonic sequence tending to zero: $\lim_{n\to\infty} a_n = 0.$

Assumption (3.44) is more general than the assumption of the convergence of series (3.43), so that Abel's test is a consequence of Dirichlet's test.

7. Infinite products and their convergence

Suppose we have a sequence of numbers (or functions)

$$p_1, p_2, p_3, \dots, p_n, \dots . \tag{3.45}$$

The symbol

$$\prod_{n=1}^{\infty} p_n = p_1 p_2 p_3 \cdots p_n \cdots \tag{3.46}$$

s known as an *infinite product.*

The product of a finite number of consecutive terms

$$P_1 = p_1, \; P_2 = p_1 p_2, \; P_3 = p_1 p_2 p_3, \; \dots, \; P_n = p_1 p_2 \cdots p_{n-1} p_n \tag{3.47}$$

is called a *partial product.* The sequence of partial products will be denoted by $\{P_n\}$.

EXAMPLE 19.

$$\prod_{n=1}^{\infty} \frac{n(n+3)}{(n+1)(n+2)} = \frac{1.4}{2.3} \, \frac{2.5}{3.4} \, \frac{3.6}{4.5} \cdots .$$

The convergence of infinite products.

Four fundamental cases may be distinguished.

(1) The sequence of partial products $\{P_n\}$ has a finite limit different from 0:

$$\lim_{n \to \infty} P_n = P. \tag{3.48}$$

This limit is called the *value* of the product and is written as

$$P = \prod_{n=1}^{\infty} p_n. \tag{3.49}$$

The product itself is said to be convergent in this case. In Example 19,

$$P = \lim_{n \to \infty} \frac{1}{3} \, \frac{n+3}{n+1} = \frac{1}{3}.$$

(2) The sequence $\{P_n\}$ tends either to $+\infty$, or to $-\infty$.

EXAMPLE 20.

$$\prod_{n=1}^{\infty} n = 1.2.3\ldots n\ldots; \quad P_n = n!; \quad \lim_{n\to\infty} n! = \infty.$$

(3) The sequence $\{P_n\}$ has the limit 0.

EXAMPLE 21.

$$p_n = \frac{1}{2} \quad \text{for any } n, \quad n,\ P_n = \frac{1}{2^n}, \quad \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

(4) The sequence $\{P_n\}$ has no limit (e.g. oscillates).

EXAMPLE 22.

$$\prod_{n=1}^{\infty} (-1)^n \frac{n(n+3)}{(n+1)(n+2)} =$$

$$= \frac{1.4}{2.3}\left(-\frac{2.5}{3.4}\right)\frac{3.6}{4.5}\left(-\frac{4.7}{5.6}\right)\ldots, \quad -\frac{1}{3} < P_n < \frac{1}{3}.$$

In the last three cases the sequence is said to be divergent. Case (3) $\left(\lim_{n\to\infty} P_n = 0\right)$ implies a different classification to that accepted for infinite series. However, this is convenient for the statement of many theorems on infinite products.

An infinite product can be written in the form

$$\prod_{n=1}^{\infty} p_n = P_m \pi_m, \tag{3.50}$$

where

$$P_m = p_1 p_2 \cdots p_m$$

is the partial product of the first m terms, and

$$\pi_m = p_{m+1} p_{m+2} \cdots = \prod_{n=m+1}^{\infty} p_n \tag{3.51}$$

is called the *remainder product*, this being analogous to the remainder term of a series.

THEOREM 5. *If product* (3.46) *is convergent, the remainder product* (3.51) *is also convergent for any m (if all the* $p_n \neq 0$*); the convergence of the remainder product implies the convergence of the original product.*

Thus discarding a finite number of initial factors or combining at the start a finite number of factors has no effect on the convergence of an infinite product.

If an infinite product is convergent, then

$$\lim_{m\to\infty} \pi_m = 1. \tag{3.52}$$

This follows from (3.50);

$$\pi_m = \frac{P}{P_m}$$

and from the fact that $P \neq 0$.

If an infinite product is convergent, then

$$\lim_{n\to\infty} p_n = 1. \tag{3.53}$$

This follows from the chain of equations:

$$\lim_{n\to\infty} p_n = \frac{\lim P_n}{\lim P_{n-1}} = \frac{P}{P} = 1.$$

In the case of a convergent product, $p_n > 0$ as from a certain n. This follows from (3.53). In view of Theorem 5, we can assume without loss of generality that all the $p_n > 0$.

There is a connection between the convergence of infinite products and that of series.

A necessary and sufficient condition for convergence of the infinite product (3.46) *is that the series*

$$\sum_{n=1}^{\infty} \ln p_n \tag{3.54}$$

be convergent.

If S is the sum of series (3.54), we have

$$P = e^S. \tag{3.55}$$

On writing s_n for the partial sum of series (3.54), we have

$$s_n = \ln P_n, \quad P_n = e^{s_n}. \tag{3.56}$$

It follows from the continuity of the logarithmic and exponential functions that, when P_n tends to a finite positive limit P, the partial sum s_n tends to $\ln P$, and conversely, if the finite limit S exists, the limit for P is equal to e^S.

The nth term of an infinite product may be conveniently written as

$$p_n = 1+a_n$$

and product (3.46) written as

$$\prod_{n=1}^{\infty} (1+a_n), \tag{3.57}$$

while series (3.54) is written as

$$\sum_{n=1}^{\infty} \ln(1+a_n). \tag{3.58}$$

THEOREM 6. *If, for sufficiently large n, all the $a_n > 0$ (or $a_n < 0$), the necessary and sufficient condition for convergence of product* (3.57) *is that the series*

$$\sum_{n=1}^{\infty} a_n. \tag{3.59}$$

be convergent.

A necessary condition for convergence of (3.57) and (3.58) is that

$$\lim_{n\to\infty} a_n = 0, \qquad \text{whence} \qquad \lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = 1.$$

In the general case $a_n \gtrless 0$, the infinite product (3.57) is convergent if the series

$$\sum_{n=1}^{\infty} a_n^2. \tag{3.60}$$

is convergent along with series (3.59).

The product of Example 19 can be written as

$$\prod_{n=1}^{\infty} \frac{n(n+3)}{(n+1)(n+2)} = \prod_{n=1}^{\infty} \left(1-\frac{2}{(n+1)(n+2)}\right).$$

Here

$$a_n = \frac{-2}{(n+1)(n+2)}$$

and

$$\lim_{n\to\infty} a_n = 0.$$

The following will make it clear why, in the case

$$P = 0,$$

the infinite product is classified as divergent.

The necessary and sufficient condition for the infinite product to have a zero value is that series (3.54) *or* (3.58) *have the sum* $-\infty$.

This is the case for example when $a_n < 0$ and series (3.59) is divergent, or, when series (3.59) is convergent, but series (3.60) is divergent.

Product (3.46) is said to be *absolutely* convergent when series (3.54) of the logarithms of its factors is absolutely convergent.

An absolutely convergent product has the *commutative* property. *The necessary and sufficient condition for absolute convergence of product* (3.57) *is that series* (3.59) *be absolutely convergent.*

We have, for Example 19:

$$|a_n| = \frac{2}{(n+l)(n+2)} = 2\frac{1}{(n+1)(n+2)} < 2\frac{1}{n^2}.$$

But $\sum\limits_{n=1}^{\infty} 1/n^2$ is a convergent series (see example 15), so that the product is absolutely convergent.

It may be remarked that the theory of functional products bears the same relationship to the theory of numerical products as the theory of functional series to that of numerical series. Let us give an example of a functional product.

EXAMPLE 23. The product

$$x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)$$

represents $\sin x$ for any x.

8. Double series. Fundamental concepts and definitions

In addition to ordinary (simple) infinite series, multiple series are employed in analysis and applied mathematics, e.g. double, triple, etc., series. We shall confine ourselves here to a brief outline of the theory of double series.

Definition. *A double series is defined by the symbol*

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{kl} \quad \left(or \quad \sum_{k,\,l=0}^{\infty} a_{kl}\right), \tag{3.61}$$

and is made up of the terms of the double sequence $\{a_{kl}\}$ ($k = 0, 1, 2, 3, \ldots;\ l = 0, 1, 2, 3, \ldots$); *corresponding to it, we have a double sequence* $\{s_{mn}\}$ *of partial sums*

$$s_{mn} = \sum_{k,\,l=0}^{k=m-1,\,l=n-1} a_{kl}. \tag{3.62}$$

If a finite limit exists on simultaneous and independent increase of the indices m and n,

$$S = \lim_{\substack{n\to\infty\\ m\to\infty}} s_{mn} = \sum_{k,\,l=0}^{\infty} a_{kl}, \tag{3.63}$$

the limit is called the sum of the double series (3.61), and the series is said to be *convergent* in this case; otherwise it is *divergent*.

If the terms a_{kl} are summed consecutively, first over one index, then over the other, the double sum will be described as an *iterated series*. Obviously, the following two cases are possible here:

$$\sum_{l=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{kl}\right) \tag{3.64}$$

or

$$\sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty} a_{kl}\right). \tag{3.65}$$

Iterated series (3.64) is said to be *convergent* if the series

$$A_l = \sum_{k=0}^{\infty} a_{k_l} \tag{3.66}$$

(for any fixed subscript l) and

$$\sum_{l=0}^{\infty} A_l \tag{3.67}$$

are convergent.

Similarly, series (3.65) is said to be *convergent* if the series

$$B_k = \sum_{l=0}^{\infty} a_{kl} \tag{3.68}$$

(with any fixed subscript k) and

$$\sum_{k=0}^{\infty} B_k \tag{3.69}$$

are convergent.

Let us introduce the notation

$$A = \lim_{n\to\infty}\left(\lim_{m\to\infty} s_{mn}\right). \tag{3.70}$$

Similarly,

$$B = \lim_{m\to\infty}\left(\lim_{n\to\infty} s_{mn}\right). \tag{3.71}$$

A series for which all the $a_{kl} \geqq 0$ is called a *double series with positive terms.*

If the double series $\sum_{k,l=0}^{\infty} a_{kl}$ and $\sum_{k,l=0}^{\infty} |a_{kl}|$ are simultaneously convergent, the double series is said to be *absolutely convergent.* Whereas if $\sum_{k,l=0}^{\infty} a_{kl}$ is convergent, but $\sum_{k,l=0}^{\infty} |a_{kl}|$ is divergent, the double series $\sum_{k,l=0}^{\infty} a_{kl}$ is *non-absolutely* (or *conditionally*) *convergent.*

9. Some properties of double series

I. *If double series* (3.61) *and series* (3.66), (3.67) *are convergent, the iterated series* (3.64) *is also convergent and has the same as the double series.*

Similarly, the following property holds:

II. *If double series* (3.61) *and series* (3.68), (3.69) *are convergent, iterated series* (3.65) *is also convergent, and has the same sum as the double series.*

REMARK. Generally speaking, the convergence of series (3.66) and (3.68) does not follow from the convergence of double series (3.61)

III. *The necessary and sufficient condition for the convergence of the double series* (3.61) *with positive terms is that its partial sums be bounded.*

IV. *If one of the three series* (3.61), (3.64) *and* (3.65) *with positive* *terms is convergent, the other two are convergent and have the same sum.*

A one-to-one correspondence can be established between pairs of equal terms of the double sequence $\{a_{kl}\}$ and the ordinary sequence $\{b_s\}$. We shall say for brevity in this case that the double series $\sum\limits_{k,\,l=0}^{\infty} a_{kl}$ and the series $\sum\limits_{s=0}^{\infty} b_s$ consist of the same terms.

V. *If the double series* $\sum\limits_{k,\,l=0}^{\infty} a_{kl}$ *and the series* $\sum\limits_{s=0}^{\infty} b_s$ *consist of the same terms, the convergence of one implies the convergence of the other, and they have the same sum.*

VI. *If the double series* $\sum\limits_{k,\,l=0}^{\infty} |a_{kl}|$ *is convergent, the series* $\sum\limits_{k,\,l=0}^{\infty} a_{kl}$ *is convergent (but the converse does not hold).*

VII. *If the double series* $\sum\limits_{k,\,l=0}^{\infty} a_{kl}$ *and the series* $\sum\limits_{s=0}^{\infty} b_s$ *consist of the same terms, the absolute convergence of one implies the absolute convergence of the other. Both series now have the same sum.*

VIII. *The terms of an absolutely convergent series can be rearranged in any manner without changing the sum.*

The above properties can be useful in proving the convergence, and computing the sum of a double series.

Example 24. It is easily shown that the double series

$$\sum_{k,\,l=0}^{\infty} \frac{1}{(k+l)^{\alpha}} \tag{3.72}$$

is convergent for $\alpha > 2$ and divergent for $\alpha \leqq 2$.

For, on putting $k+l = m$, we find that $m-1$ terms of series (3.72) are equal to $1/m^{\alpha}$. Series (3.72) can therefore be written as the simple series

$$\sum_{m=2}^{\infty} \frac{m-1}{m^{\alpha}} = \sum_{m=2}^{\infty} \frac{1}{m^{\alpha-1}} - \sum_{m=2}^{\infty} \frac{1}{m^{\alpha}}\,. \tag{3.73}$$

Both the series on the right-hand side of (3.73) are convergent for $\alpha > 2$ (Example 15), so that the series on the left-hand side is convergent. By Property V of double series, the double series (3.72) is also convergent, and its sum is equal to the sum of the simple series (3.73).

10. Some convergence tests for double series of positive terms. Estimates of remainder term

A necessary condition for the convergence of any double series $\sum\limits_{k,\,l=0} a_{kl}$ *is that*

$$\lim_{\substack{k\to\infty\\ l\to\infty}} a_{kl} = 0, \tag{3.74}$$

when k and l tend to infinity independently of each other.

A stricter necessary condition than (3.74) can be given for the convergence of double series with positive terms.

For a double series with positive terms to be convergent, it is necessary that

$$\lim_{k\to\infty} \sum_{l=0}^{\infty} a_{kl} = 0 \tag{3.75}$$

and

$$\lim_{l\to\infty} \sum_{k=0}^{\infty} a_{kl} = 0, \tag{3.76}$$

so that all the more, the following conditions must be satisfied:

$$\lim_{k\to\infty} \sum_{l=0}^{k} a_{kl} = 0 \tag{3.77}$$

and

$$\lim_{l\to\infty} \sum_{k=0}^{l} a_{kl} = 0. \tag{3.78}$$

From (3.77) and (3.78), we see that the following conditions must also be satisfied:

$$\left.\begin{aligned} &\lim_{\substack{k\to\infty\\ l\to\infty}} a_{kl} = 0, && \\ &\lim_{k\to\infty} a_{kl} = 0 && \text{for any } l, \\ &\lim_{l\to\infty} a_{kl} = 0 && \text{for any } k. \end{aligned}\right\} \tag{3.79}$$

As in the case of ordinary series with positive terms (see § 1, sec. 3–5), comparison of the terms of two series leads to a number of general tests for the convergence of double series with positive terms, and corresponding remainder term estimates can be given.

Suppose we have the double series

$$\sum_{k,\,l=0}^{\infty} a_{kl} \tag{3.80}$$

with positive terms, and some other series

$$\sum_{k,\,l=0}^{\infty} b_{kl}. \tag{3.81}$$

We assume that the convergence or the sum of the series (3.81) is known to us in advance. We introduce the notation: $k+l = n$,

$$\varliminf_{n\to\infty} \frac{b_{kl}}{a_{kl}} = \underline{A}, \tag{3.82}$$

$$\varlimsup_{n\to\infty} \frac{b_{kl}}{a_{kl}} = \bar{A}. \tag{3.83}$$

In the case when $\underline{A} = \bar{A}$, we have

$$\lim_{n\to\infty} \frac{b_{kl}}{a_{kl}} = A. \tag{3.84}$$

The remainder term of double series (3.80) will be defined by

$$R_{mm} = r_{m0} + r_{0m} + r_{mm}, \tag{3.85}$$

where

$$r_{m0} = \sum_{k=m,\,l=0}^{k=\infty,\,l=m-1} a_{kl}, \qquad r_{0m} = \sum_{k=0,\,l=m}^{k=m-1,\,l=\infty} a_{kl}$$

and

$$r_{mm} = \sum_{k,\,l=m}^{\infty} a_{kl}.$$

We shall write B_{mm} for expression (3.85), corresponding to the remainder term of series (3.81).

Using the above notation, the following general sufficiency tests can be stated for the convergence of the double series (3.80):

1°. *If* $0 \leqq B_{mm} < +\infty$ *for all* $m > N$ *and* $\underline{A} > 0$, *the series is convergent.*

2°. *If* $-\infty < B_{mm} \leqq 0$ *for all* $m > N$ *and* $\bar{A} < 0$, *the series is convergent.*

3°. *If, with finite* $\underline{A}$ *and* $\bar{A}$ *of the same sign,* $B_{mm} = \pm\infty$, *the series is divergent.*

4°. *If* $0 < B_{mm} < +\infty$ *for all* $m > N$, *and the conditions* $-\infty < \underline{A} < +\infty$ *and* $\overline{A} = +\infty$ *are satisfied, the series is convergent.*

5°. *If* $-\infty < B_{mm} < 0$ *for* $m > N$ *and the conditions* $-\infty < \overline{A} < 0$ and $\underline{A} = -\infty$ *are satisfied, the series is convergent.*

When series (3.80) is convergent, the following estimates hold for the remainder term:

$$\frac{B_{mm}}{\inf\limits_{k+l \geqq m} \dfrac{b_{kl}}{a_{kl}}} \substack{\geqq \\ \leqq} R_{mm} \substack{\geqq \\ \leqq} \frac{B_{mm}}{\sup\limits_{k+l \geqq m} \dfrac{b_{kl}}{a_{kl}}}. \tag{3.86}$$

When $\overline{A} = +\infty$ or $\underline{A} = -\infty$ (or $A = \pm\infty$),

$$R_{mm} \leqq \frac{|B_{mm}|}{\inf\limits_{k+l \geqq m} \left| \dfrac{b_{kl}}{a_{kl}} \right|}. \tag{3.87}$$

Further, on choosing the b_{kl} by different methods, we can obtain from the above general convergence tests various practical tests which are sufficient for the convergence of a double series with positive terms. Inequalities (3.86) and (3.87) enable us to give remainder term estimates corresponding to the convergence test chosen.

We shall confine ourselves here to one concrete case, e.g. by putting

$$b_{kl} = a_{k+1,\,l+1} - a_{k+1,\,l} - a_{k,\,l+1} + a_{kl}. \tag{3.88}$$

It now follows from the above general tests that the double series $\sum\limits_{k,\,l=0}^{\infty} a_{kl}$ is convergent if:

(1)
$$\lim_{p \to \infty} a_{pp} = 0; \tag{3.89}$$

(2)
$$\underline{A} = \underline{\lim}_{k+l \to \infty} \frac{b_{kl}}{a_{kl}} > 0; \tag{3.90}$$

and is divergent if:

$$\overline{A} = \overline{\lim}_{k+l \to \infty} \frac{b_{kl}}{a_{kl}} < 0. \tag{3.91}$$

The remainder term estimate corresponding to this test has the form

$$\frac{a_{m0}+a_{0m}-a_{mm}}{\sup\limits_{k+l\geqq m}\dfrac{b_{kl}}{a_{kl}}} \leqq R_{mm} \leqq \frac{a_{m0}+a_{0m}-a_{mm}}{\inf\limits_{k+l\geqq m}\dfrac{b_{kl}}{a_{kl}}}, \tag{3.92}$$

where b_{kl} is given by equation (3.88).

Fulfilment of condition (3.90) only is not sufficient for convergence of the double series.

EXAMPLE 25. The series

$$S = \sum_{k,l=0}^{\infty} \left(\frac{3}{2}\right)^k \left(\frac{6}{5}\right)^l,$$

is obviously seen to be divergent; at the same time, the condition $\underline{A} = 0{\cdot}1 > 0$ is satisfied although $\lim\limits_{p\to\infty} a_{pp} = \infty$.

EXAMPLE 26. Let us take the double series

$$S = \sum_{k,l=0}^{\infty} \frac{1}{(k+1)!\,(l+1)!}\,. \tag{3.93}$$

Condition (3.89) is satisfied here, since

$$\lim_{p\to\infty} a_{pp} = \lim_{p\to\infty} \frac{1}{[(p+1)!]^2} = 0.$$

We have by (3.90):

$$\lim_{k+l\to\infty} \frac{b_{kl}}{a_{kl}} = \lim_{k+l\to\infty} \frac{a_{k+1,\,l+1}-a_{k,\,l+1}-a_{k+1,\,l}+a_{kl}}{a_{kl}} =$$

$$= \lim_{k+l\to\infty} \frac{(k+1)(l+1)}{(k+2)(l+2)} = \begin{cases} \dfrac{l+1}{l+2} & \text{for fixed } l; \\ \dfrac{k+1}{k+2} & \text{for fixed } k; \\ 1, & \text{if simultaneously } k \text{ and } l \to \infty. \end{cases}$$

But obviously, $\frac{1}{2} \leqq (l+1)/(l+2) \leqq 1$ and $\frac{1}{2} \leqq (k+1)/(k+2) \leqq 1$, i.e. $\underline{A} = \frac{1}{2}$, for all k, $l \geqq 0$. Series (3.93) is thus convergent.

To estimate R_{mm} in accordance with (3.92), we first observe that, for the function $F(x, y) = (x+1)(y+1)/(x+2)(y+2)$, necessary conditions for an extremum: $\partial F/\partial x = \partial F/\partial y = 0$, are not satisfied for any finite positive x and y, i.e. the greatest and least values of the function can only be attained on the boundary of the domain $m \leqq x,\ y < \infty$.

It may easily be shown by using this fact that

$$\inf_{k+l \geqq m} \frac{b_{kl}}{a_{kl}} = \inf_{k+l \geqq m} \frac{(k+1)(l+1)}{(k+2)(l+2)} = \frac{m+1}{m+2} \cdot \frac{1}{2}$$

and

$$\sup_{k+l \geqq m} \frac{(k+1)(l+1)}{(k+2)(l+2)} = 1.$$

On substituting the values obtained in inequality (3.92), we get

$$\frac{2}{(m+1)!} - \frac{1}{[(m+1)!]^2} \leqq R_{mm} \leqq \frac{4(m+2)}{(m+1)\,(m+1)!} - \frac{2(m+2)}{(m+1)\,[(m+1)!]^2}\,.$$

For instance, we obtain with $m = 4$:

$$0{\cdot}0159 \leqq R_{mm} \leqq 0{\cdot}0382.$$

§ 2. Series of functions

1. Fundamental properties and convergence tests

We shall discuss functional series in this article, i.e. series whose terms are functions. We shall confine ourselves simplicity to the series

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots = \sum_{n=1}^{\infty} u_n(x), \qquad (3.94)$$

the terms of which are functions of a single variable. Immediate generalization is possible to the case of two or more variables.

DEFINITION. *The expression* (3.94) *is called a functional series; it is formed from the terms of the function sequence* $\{u_n(x)\}$, *defined on the set* $X = \{x\}$ *of the numerical axis* E_1, *and has a corresponding sequence* $\{s_n\}$:

$$s_n = \sum_{k=1}^{n} u_k(x)$$

of partial sums.

There are different types of convergence for the function $s_n(x)$: *uniform*, *non-uniform*, *in the mean*, etc. Corresponding to these we have different forms of convergence of the functional series.

Let the sequence $\{s_n(x)\}$ be *uniformly* (*non-uniformly*) *convergent* on X to the function $S(x)$. Series (3.94) is then said to be *uniformly* (*non-uniformly*) *convergent*, and $S(x)$ is its sum:

$$S(x) = \sum_{n=1}^{\infty} u_n(x). \tag{3.95}$$

A Condition for Uniform Convergence of a Series: The necessary and sufficient condition for series (3·94) *to be uniformly convergent on X is that, given any $\varepsilon > 0$, there exists an N independent of x such that, given any $n > N$ and any $m = 1, 2, 3, \ldots$, the inequality*

$$\left| \sum_{k=n+1}^{n+m} u_k(x) \right| < \varepsilon$$

holds for all $x \in X$.

If all the terms of series (3.94), uniformly convergent on the set X, are multiplied by the same function $v(x)$, bounded on X,

$$|v(x)| \leqq M,$$

the uniform convergence is preserved.

Tests for uniform convergence of series.

1°. Weierstrass's Test. *If the terms of series* (3.94) *satisfy on the set X the inequalities*

$$|u_n(x)| \leqq c_n \qquad (n = 1, 2, 3, \ldots), \tag{3.96}$$

where c_n are the terms of a convergent numerical series, (3.94) *is uniformly convergent on X.*

When (3·96) holds, the series

$$\sum_{n=1}^{\infty} c_n$$

is described as *majorant* for series (3.94). A functional series satisfying Weierstrass's test is *absolutely* convergent, and moreover, the series

$$\sum_{n=1}^{\infty} |u_n(x)|.$$

is uniformly convergent.

Cases are possible when series (3·94) is uniformly but not absolutely convergent.

The following tests hold for functional series of the form

$$\sum_{n=1}^{\infty} a_n(x)b_n(x) = a_1(x)b_1(x) + \ldots + a_n(x)b_n(x) + \ldots . \tag{3.97}$$

2°. ABEL'S TEST. *Series* (3·97) *is uniformly convergent on a set X if the series*

$$\sum_{n=1}^{\infty} b_n(x) = b_1(x) + b_2(x) + \ldots \tag{3.98}$$

is uniformly convergent on the set X, while the functions $a_n(x)$ *form a monotonic sequence for any x, and are bounded for any x and n:*

$$|a_n(x)| \leqq K.$$

3°. DIRICHLET'S TEST. *Series* (3·97) *is uniformly convergent on the set X if the partial sums* $s_n(x)$ *of series* (3·98) *are bounded for all x and n:*

$$|s_n(x)| \leqq M,$$

while the functions $a_n(x)$ *form, for any x, a monotonic sequence, uniformly convergent to zero on the set X.*

Let us also note some properties of the sum of a functional series. If the functions $u_n(x)$ of series (3·94) are defined in the interval $X = (a, b)$ and are all continuous at the point $x = x_0$ of this interval, and in addition, series (3·94) is uniformly convergent, the sum $S(x)$ of the series is also continuous at the point $x = x_0$.

The following proposition is a consequence of this: if the functions $u_n(x)$ are continuous throughout the interval $X = (a, b)$ and if the series is uniformly convergent there, the sum $S(x)$ of the series is continuous throughout the interval. The uniform convergence in these propositions is a *sufficient* but not a necessary condition, since there are series, non-uniformly convergent in an interval, having a sum continuous in the interval. We shall not state here necessary and sufficient conditions for continuity of the sum of a series, and shall only mention that uniform convergence is necessary and sufficient provided the terms of the series are in addition positive in the interval.

Term by term integration of series. If the terms $u_n(x)$ of series (3·94) are continuous in the interval $X = (a, b)$ and the series is uniformly convergent in this interval, the integral of the sum $S(x)$ of the series is equal to the sum of the integrals of the terms:

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx. \tag{3.99}$$

In other words, *term by term integration of the series* is permissible under these conditions.

The following is a generalization of the above theorem: if the terms $u_n(x)$ of the series are integrable (in Riemann's sense) in the interval $X = (a, b)$ while the series is uniformly convergent, the sum $S(x)$ of the series will also be integrable and (3·99) will hold.

Term by term differentiation of series. If the terms $u_n(x)$ of series (3.84) are defined in an interval $X = (a, b)$ and have continuous derivatives $u_n'(x)$ in X, and series (3·94) is convergent in X, while in addition the series formed from the derivatives

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots,$$

is *uniformly convergent*, the sum $S(x)$ of series (3·94) has a derivative in X, and

$$S'(x) = \sum_{n=1}^{\infty} u_n'(x), \tag{3.100}$$

in other words, the derivative of the sum is equal to the sum of the series formed from the derivatives, i.e. *term by term differentiation of series* (3·94) is permissible. It may be mentioned that the restrictions of this theorem can be somewhat relaxed (see ref. 11, vol. II, § 407).

2. Power series

This branch of the theory of functional series is particularly important.

A series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + \ldots + a_n(x - x_0)^n + \ldots \tag{3.101}$$

is called a *power series*. On substituting $x_1 = x-x_0$ (we shall omit the subscript 1), it reduces to the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots, \qquad (3.102)$$

with which we shall be concerned in what follows.

There are three possible cases.

Case 1. Series (3.102) is convergent for any real x, i.e. throughout the axis E_1. In this case the series is said to be *convergent everywhere.*

Case 2. Series (3.102) is convergent only for $x = 0$. The series is now said to be *divergent everywhere.*

Case 3. The series is convergent in some interval $(-R, +R)$, $0<R< +\infty$; the number R is called the *radius of convergence* of the series.

We shall accept the convention that Case 1 corresponds to a radius of convergence $R = \infty$, and Case 2 to a radius of convergence $R = 0$.

Example 27. The series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

is convergent for any x ($R = \infty$).

Example 28. The series

$$\sum_{n=1}^{\infty} n!\,x^n = x+2!\,x^2+3!\,x^3+\ldots$$

is convergent only for $x = 0$ ($R = 0$).

Example 29. The series

$$\sum_{n=1}^{\infty} x^n = 1+x+x^2+\ldots$$

is convergent in the interval $(-1, +1)$ ($R = 1$).

Example 30. The series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots$$

is convergent in the semi-interval $(-1, +1)$ ($R = 1$).

EXAMPLE 31. The series

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2}=x+\frac{x^2}{4}+\frac{x^3}{9}+\dots$$

is convergent in the segment $[-1, +1]$ ($R = 1$).

If the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\varrho,$$

exists, (see d'Alembert's convergence test) then

$$R=\frac{1}{\varrho}. \tag{3.103}$$

($R = \infty$ for $\varrho = 0$, $R = 0$ for $\varrho = \infty$).

On starting from Cauchy's test, we get

$$R=\frac{1}{\varrho}=\frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}. \tag{3.104}$$

If series (3.102) has a radius of convergence $R > 0$, no matter what the positive number $r < R$, series (3.102) is convergent uniformly with respect to x in the segment $[-r, r]$.

The sum $S(x)$ of series (3.102) is a continuous function of x for all x between $-R$ and R.

THEOREM 7 (on the identity of power series). *If two power series*

$$\sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad \sum_{n=0}^{\infty}b_nx^n$$

have the same sum in the neighbourhood of the point $x = 0$, the series are identical, i.e. their corresponding coefficients are equal: $a_n = b_n$.

The following propositions concern the behaviour of a power series at the ends of the interval of convergence. If power series (3.102) is divergent at an end of the interval of convergence, the convergence cannot be uniform in the semi-interval $(0, R)$.

The converse is: *if the power series* (3.102) *is convergent for* $x=R$ ($x=-R$) *(even non-absolutely,) the series is uniformly convergent throughout the segment* $[0, R]$ ($[-R, 0]$).

THEOREM 8 (Abel). *If series* (3.102) *is convergent for* $x = R$, *its sum remains continuous (from the left) for this value of* x, *i.e.*

$$\lim_{x\to R-0} S(x) = S(R).$$

See Example 30 for $x = -R = -1$ and Example 31 for $x = -R = -1$, $x = R = 1$.

3. Operations on power series. Taylor series. Integration and differentiation of power series

(a) Power series (3.102) can be integrated term by term in the segment $[0, x]$, where $|x| < R$:

$$\int_0^x S(x)\,dx = \sum_{n=0}^{\infty}\int_0^x a_n x^n\,dx = \sum_{n=0}^{\infty} a_n\frac{x^{n+1}}{n+1}\,; \qquad (3.105)$$

x may reach the end of the segment at which series (3.102) is convergent.

(b) Power series (3.102) can be differentiated term by term as many times as desired inside its segment of convergence:

$$\left\{\begin{aligned} S'(x) &= \sum_{n=1}^{\infty} na_n x^{n-1},\\ S''(x) &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},\\ &\dots\dots\dots\dots\dots\dots\\ S^{(m)}(x) &= \sum_{n=m}^{\infty} n(n-1)\dots(n-m+1)a_n x^{n-m}; \end{aligned}\right.$$

this also holds for an end point of the segment at which the series is convergent.

Taylor series. Different forms of the remainder term. A function expressible as a power series in its interval of convergence has derivatives of all orders inside the interval. The series itself is the *Taylor series* of the function.

If the function is expanded as a Taylor series in the neighbourhood of the point x_0,

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \tag{3.106}$$

in this neighbourhood, where

$$a_n = \frac{f^{(n)}(x_0)}{n!} \qquad (0! = 1). \tag{3.107}$$

The particular case of a Taylor series when $x_0 = 0$ is sometimes called a *Maclaurin series*. As follows from (3.106) and (3.107), it has the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{3.108}$$

where

$$a_n = \frac{f^{(n)}(0)}{n!}\,.$$

On confining ourselves to a finite number of terms in a Taylor series, a function $f(x)$ can be written as a partial sum s_n of the series plus a remainder term r_n:

$$f(x) = \sum_{k=0}^{n} a_k(x-x_0)^k + \sum_{k=n+1}^{\infty} a_k(x-x_0) = s_n(x)+r_n(x). \tag{3.109}$$

When $x \to x_0$, the remainder term $r_n(x)$ is an infinitesimal of order higher than n (compared with $x-x_0$). There are various familiar forms of the remainder term of a Taylor series.

The form of Schlömilch and Roche:

$$r_n(x) = \frac{f^{(n+1)}[x_0+\theta(x-x_0)]}{n!\,p}(1-\theta)^{n+1-p}(x-x_0)^{n+1}. \tag{3.110}$$

Here $p > 0$, $0 < \theta < 1$

On assigning concrete values to p, we get more specialized forms of the remainder term:

Lagrange's form ($p = n+1$):

$$r_n(x) = \frac{f^{(n+1)}[x_0+\theta(x-x_0)]}{(n+1)!}(x-x_0)^{n+1} \qquad (0 < \theta < 1). \tag{3.111}$$

Cauchy's form $(p = 1)$:

$$r_n(x) = \frac{f^{(n+1)}[x_0+\theta(x-x_0)]}{n!}(1-\theta)^n(x-x_0)^{n+1} \quad (0<\theta<1). \tag{3.112}$$

In spite of the indeterminate size of θ, these forms of the remainder term enable us to estimate the accuracy of replacing $f(x)$ by the nth degree polynomial $s_n(x)$. If the $(n+1)$th derivative is bounded in absolute value in the interval between x and x_0 by the number M, we have from (3.111):

$$|r_n(x)| \leqq \frac{M|x-x_0|^{n+1}}{(n+1)!}.$$

Example 32. The Taylor expansion of the function $f(x) = e^x$ in the neighbourhood of the point $x_0 = 0$ is

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+r_n(x),$$

where

$$r_n(x) = \frac{x^{n+1}}{(n+1)!}+\frac{x^{n+2}}{(n+2)!}+\dots.$$

By (3.111):

$$r_n(x) = \frac{e^{\theta x}}{(n+1)!}x^{n+1}.$$

Here (when $x > 0$):

$$|r_n(x)| < e^x\frac{x^{n+1}}{(n+1)!}.$$

For instance, when $x = 1$:

$$|r_n(1)| < \frac{3}{(n+1)!}.$$

The remainder term can also be expressed in the *integral form*, which contains no indeterminate numbers:

$$r_n(x) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt. \tag{3.113}$$

We can write, in the above example:

$$r_n(x) = \frac{1}{n!}\int_0^x e^t(x-t)^n\,dt.$$

Notice that a Taylor series may be convergent, and not represent its generating function $f(x)$.

EXAMPLE 33 (Cauchy). Let $\varphi(x)$ be a given function, expressible in the neighbourhood of $x = 0$ by the series

$$\varphi(x) = \varphi(0) + \frac{\varphi'(0)}{1!}x + \frac{\varphi''(0)}{2!}x^2 + \dots . \tag{3.114}$$

We add to it the function $\psi(x) = e^{-1/x^2}$. If we complete the definition of $\psi(x)$ by putting $\psi(0) = 0$, both $\psi(0)$ and all its derivatives $\psi^{(n)}(0)$ vanish, so that the Taylor expansion of $\psi(x)$ in the neighbourhood of $x = 0$ is identically zero. But now:

$$f(x) = \varphi(x) + \psi(x) = \varphi(0) + \frac{\varphi'(0)}{1!}x + \frac{\varphi''(0)}{2!}x^2 + \dots . \tag{3.115}$$

It is clear from this that equality of the expansions of the functions (3.114) and (3.115) does not allow us to deduce the equality of the left-hand sides: $f(x) \neq \varphi(x)$.

In general, if two functions $f(x)$ and $\varphi(x)$ are equal at $x = x_0$ and all their derivatives are equal, while at the same time the functions are not identically equal, their Taylor expansions are nevertheless the same in the neighbourhood of $x = x_0$.

Various operations on power series. Substitution of a series into a series. Suppose we have the series

$$z = f(y) = a_0 + a_1y + a_2y^2 + \dots = \sum_{n=0}^{\infty} a_ny^n, \tag{3.116}$$

convergent in the interval $(-R, R)$. The variable y is expressible in turn as a power series in x:

$$y = \varphi(x) = b_0 + b_1x + b_2x^2 + \dots = \sum_{n=0}^{\infty} b_nx^n, \tag{3.117}$$

convergent in the interval $(-r, r)$. Say we want to express z as a power series in x and to find the interval of convergence of the series.

On formally substituting series (3.117) into series (3.116), we get

$$z = \sum_{n=0}^{\infty} A_n x^n, \tag{3.118}$$

where

$$\left.\begin{aligned} A_0 &= a_0 + a_1 b_0 + a_2 b_0^2 + \ldots, \\ A_1 &= a_1 b_1 + 2a_2 b_0 b_1 + 3a_3 b_0^2 b_1 + \ldots, \\ A_2 &= a_1 b_2 + a_2(b_1^2 + 2b_0 b_2) + 2a_3(b_0 b_1^2 + b_0^2 b_2) + \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}$$

The question of the convergence of series (3.118) is answered by

THEOREM 9. 1°. *If*

$$|b_0| > R,$$

series (3.118) *is divergent, whereas, if*

$$|b_0| < R,$$

the series is convergent in the interval $(-R_1, R_1)$, *where*

$$R_1 = \frac{(R - |b_0|)\varrho}{M + R - |b_0|}, \tag{3.119}$$

ϱ *being an arbitrary positive number that satisfies the condition* $\varrho < r$ *and can be taken as close as desired to* r, *while* M *is the least upper bound of the numbers* $|b_m|\varrho^m$ $(m = 1, 2, 3, \ldots)$, *so that* $|b_m|\varrho^m \leqq M$ *or all* m.

2°. *If* $b_0 = 0$, *series* (3.118) *will be convergent in an interval* $(-R_1, R_1)$, *where*

$$R_1 = \frac{R\varrho}{M+R}.$$

3°. *If the series* $z = \sum a_n y^n$ *is convergent for all* y, *i.e. in the interval* $(-\infty, +\infty)$, *series* (3.118) *will be convergent for* $|x| < r$, *i.e. in the interval* $(-r, r)$.

Multiplication and division of power series.

THEOREM 10. *The product of the series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad \varphi(x) = \sum_{n=0}^{\infty} b_n x^n,$$

convergent respectively in the intervals

$$I_1 = (-R_1,\ R_1) \quad \textit{and} \quad I_2 = (-R_2,\ R_2),$$

is given by the equation

$$F(x) = f(x)\varphi(x) = c_0 + c_1x + c_2x^2 + \ldots = \sum_{n=0}^{\infty} c_nx^n$$

where $c_0 = a_0b_0$, $c_1 = a_0b_1 + a_1b_0$, $c_2 = a_0b_2 + a_1b_1 + a_2b_0$, *etc., and is convergent in the smaller of the intervals* I_1, I_2.

The quotient of the division of 1 by the power series $1 + \sum_{n=1}^{\infty} b_nx^n$:

$$f(x) = \frac{1}{1 + \sum\limits_{n=1}^{\infty} b_nx^n},$$

convergent in an interval $(-r, r)$, is given by the series

$$f(x) = 1 + A_1x + A_2x^2 + \ldots = 1 + \sum_{n=1}^{\infty} A_nx^n, \qquad (3.120)$$

where $A_1 = -b_1$, $A_2 = b_1^2 - b_2$, $A_3 = -b_1^3 + 2b_1b_2 - b_3$, etc.

THEOREM 11. *Series* (3.120) *is convergent in the interval* $(-\varrho/[M+1],\ \varrho/[M+1])$, *where* $0 < \varrho < r$ *and* ϱ *can be taken as close as desired to* r, *while* M *is the least upper bound of the numbers* $|b_m|\varrho^m$ $(m = 1, 2, 3, \ldots)$.

We can easily pass from this to the division of one power series by another:

$$\frac{\varphi(x)}{\psi(x)} = \frac{a_0 + a_1x + a_2x^2 + \ldots}{b_0 + b_1x + b_2x^2 + \ldots} = \frac{\sum\limits_{n=0}^{\infty} a_nx^n}{\sum\limits_{n=0}^{\infty} b_nx^n}.$$

We have

$$\frac{\varphi(x)}{\psi(x)} = \frac{1}{b_0}(a_0 + a_1x + a_2x^2 + \ldots)(1 + A_1x + A_2x^2 + \ldots),$$

where

$$1 + \sum_{n=1}^{\infty} A_nx^n = \frac{1}{1 + \frac{1}{b_0}\sum\limits_{1}^{\infty} b_nx^n}$$

is obtained as indicated above. The expansion

$$\frac{\varphi(x)}{\psi(x)} = \sum_{n=0}^{\infty} c_n x^n$$

is now convergent in an interval determined by the above theorems.

4. Complex series

A *numerical* (or *functional*) *complex series*

$$\sum_{n=1}^{\infty} c_n \tag{3.121}$$

is one in which the terms are complex numbers (or functions):

$$c_n = a_n + ib_n\,. \tag{3.122}$$

The *convergence of complex series* (3·121) to the sum $S = A+iB$ is equivalent to the convergence of two real series

$$\sum_{n=1}^{\infty} a_n, \tag{3.123}$$

$$\sum_{n=1}^{\infty} b_n, \tag{3.124}$$

to the sums A and B respectively.

We have:

Theorem 12. *If the positive series*

$$\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2}, \tag{3.125}$$

composed of the moduli of the terms of series (3·121), *is convergent, series* (3·121) *is also convergent.*

In this case series (3·121) is said to be *absolutely convergent.* D'Alembert's and Cauchy's tests retain their validity for complex series. The theorem on rearrangement of the terms of a series and the rule for term by term multiplication of series can be carried over of absolutely convergent complex series. All the theorems on absolutely convergent real series retain their validity for absolutely convergent complex series.

Functions of a complex variable. If, for every value of a complex variable z from a domain Z in the complex plane, there is a corresponding single value of another complex variable $w=u+iv$, w is called a *complex function* of z in the domain Z and we write

$$w = f(z). \tag{3.126}$$

If the function $f(z)$ is expressible by a Taylor series in the neighbourhood of the point z_0, i.e. can be expanded as a convergent power series in powers of $(z-z_0)$, it is called an *analytic function of z at the point z_0*. If $f(z)$ is expressible as a power series in the neighbourhood of any point of the domain Z (open or closed), it is said to be an *analytic function of z in the domain Z*. Functions having this property are of the greatest interest in applied mathematics.

The complex power series

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n \tag{3.127}$$

has a number of properties similar to those of real power series. We can consider in future, without loss of generality, the series

$$\sum_{n=0}^{\infty} c_n z^n. \tag{3.128}$$

Since this series has a *circle of convergence* instead of an *interval of convergence*, the term radius of convergence acquires a strict meaning here.

If the coefficients c_n of power series (3.127) are real numbers, the radius R of the *circle of convergence* coincides with the previous radius of convergence.

The following proposition holds: if the series (3.128) is convergent at some point z_0 of the circumference $|z|=R$, then as the point z approaches the point z_0 from inside the circle along a radius, we have

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z_0^n.$$

A complex power series can be differentiated term by term inside its circle of convergence.

If a function is expanded, as a series in powers of z, the distance from the origin ($z = 0$) to the nearest singular[†] point of the function is equal to the radius of convergence of the sum of the series.

EXAMPLE 34.

$$\frac{1}{1+z^2} = 1-z^2+z^4-z^6+\ldots+(-1)^n z^{2n}+\ldots = \sum_{n=0}^{\infty}(-1)^n z^{2n}. \tag{3.129}$$

If we take the real axis $z = x$, the expansion

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\ldots \tag{3.130}$$

has a radius of convergence $R = 1$, although the function $1/(1+x^2)$ and its derivatives have no discontinuities on passing through the points ± 1. On returning to expansion (3.129) in the complex plane, we see that the function $1/(1+z^2)$ has discontinuities at the points $\pm i$. This is in fact the reason for the divergence of expansion (3.130) for $|x| > 1$.

If we introduce the notation $c_n = a_n+ib_n$, (a_n, b_n are real numbers), $|z| = r$, $\arg z = \theta$, series (3.128) can be written in the form:

$$\left\{\begin{aligned} \sum_{n=0}^{\infty} c_n z^n &= \sum_{n=0}^{\infty}(a_n+ib_n)r^n e^{in\theta} = \\ &= \sum_{n=0}^{\infty} r^n(a_n+ib_n)(\cos n\theta+i\sin n\theta) = \\ &= \sum_{n=0}^{\infty} r^n(a_n\cos n\theta-b_n\sin n\theta)+i\sum_{n=0}^{\infty} r^n(b_n\cos n\theta+a_n\sin n\theta). \end{aligned}\right.$$

Further, we have, on writing $a_n r^n = A_n$, $-b_n r^n = B_n$:

$$\sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty}(A_n\cos n\theta+B_n\sin n\theta)+i\sum_{n=0}^{\infty}(-B_n\cos n\theta+A_n\sin n\theta).$$

The real and imaginary parts of the series are thus written as trigonometric series.

[†] A *singular* point is one in the neighbourhood of which a function is not expressible as a power series.

EXAMPLE 35.

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (r=|z|<1);$$

$$\frac{1}{1-z}=\frac{1}{1-(r\cos\theta+ir\sin\theta)}=\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}+$$

$$+i\frac{r\sin\theta}{1-2r\cos\theta+r^2}=\sum_{k=0}^{\infty} r^k\cos k\theta+i\sum_{k=1}^{\infty} r^k\sin k\theta,$$

from which it follows that

$$\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}=\sum_{k=0}^{\infty} r^k\cos k\theta,$$

$$\frac{r\sin\theta}{1-2r\cos\theta+r^2}=\sum_{k=1}^{\infty} r^k\sin k\theta.$$

5. Trigonometric Fourier series

A function $f(x)$ having a *period* 2π† ($f(x)=f(x+2\pi n)$, $n=\pm 1, \pm 2, \ldots$) can be expressed, under certain restrictions to be described below, as an *infinite trigonometric Fourier series*

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{3.131}$$

The *Euler–Fourier formulae* are used to find the coefficients of series (3.131):

$$\left.\begin{aligned} a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n=0,\ 1,\ 2,\ \ldots),\\ b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n=1,\ 2,\ 3,\ldots).\end{aligned}\right\} \tag{3.132}$$

In order that the Fourier series (3.131) be convergent to the given function $f(x)$, i.e. for its sum to be equal, at every point x_0 of the interval $(0,\ 2\pi)$, to the value $f(x_0)$ at this point, the function $f(x)$ must satisfy certain conditions (see below, and Chapter IV, § 2, sec. 5).

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† If the period of $f(x)$ is equal to $2L$, it can be transformed to the period 2π by bringing in the variable $y=\pi x/L$, so that all subsequent discussions retain their generality

THEOREM 13 (Localization theorem). *The behaviour (i.e. convergence or divergence) of the Fourier series of the function $f(x)$ at some point x_0 depends exclusively on the values taken by the function in the immediate vicinity of the point x_0.*

The sum S_0 of the series (3.131) for $f(x)$ at x_0 is determined as follows:

(a) when $f(x)$ is continuous at x_0, we have

$$S_0 = f(x_0);$$

(b) when $f(x)$ has a discontinuity of the first kind at x_0 (so that the limits $f(x_0+0)$ and $f(x_0-0)$ exist), we have

$$S_0 = \frac{f(x_0+0)+f(x_0-0)}{2}.$$

We say that $f(x)$ is *smooth* in the segment $[a, b]$ if it has a *continuous* derivative in this segment.

A continuous function $f(x)$ is described as *piecewise smooth* in the segment $[a, b]$ if the segment can be split into a finite number of subsegments, on each of which $f(x)$ is smooth.

A discontinuous $f(x)$ is described as *piecewise smooth* on the segment $[a, b]$ if: (1) it has only points of discontinuity of the first kind (and only a finite number of these) on the segment, (2) on each of the subsegments $[\alpha, \beta]$ into which the original segment is split by the points of discontinuity, the continuous function

$$g(x) = \begin{cases} f(\alpha+0) & \text{for} \quad x = \alpha, \\ f(x) & \text{for} \quad \alpha < x < \beta, \\ f(\beta-0) & \text{for} \quad x = \beta \end{cases}$$

is piecewise smooth.

A convergence test for Fourier series. The Fourier series of a piecewise smooth (continuous or discontinuous) function $f(x)$ of period 2π is convergent for all values of x_0, its sum being equal to S_0.

If a piecewise smooth function $f(x)$ is continuous everywhere, its Fourier series is absolutely and uniformly convergent.

THEOREM 14. *Let $f(x)$ be an absolutely integrable function of period 2π, continuous and possessing an absolutely integrable derivative in some segment $[a, b]$ (the derivative may cease to exist at individual* *points). The Fourier series is then uniformly convergent to* $f(x)$ *in every segment* $[a+\delta, b-\delta]$ $(\delta > 0)$.

This theorem holds, in particular, for an absolutely integrable $f(x)$ of period 2π, continuous and piecewise smooth on the segment $[a, b]$.

EXAMPLE 36. The function $f(x) = -\ln\left|2\sin\frac{1}{2}x\right|$, which becomes infinite for $x = 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$, has period 2π. Its Fourier series is

$$-\ln\left|2\sin\frac{x}{2}\right| = \cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \ldots.$$

THE DIRICHLET–JORDAN TEST. *The Fourier series of a function* $f(x)$ *is convergent at* x_0 *to the sum* S_0 *if the function is of bounded variation in some segment* $[x_0-\delta, x_0+\delta]$.

A more specialized statement is:

DIRICHLET'S TEST. *If* $f(x)$ *of period* 2π *is piecewise monotonic on a segment* $[-\pi, \pi]$ *and has a finite number of discontinuities there, then its Fourier series is convergent to the sum* $f(x_0)$ *at every point of continuity and to the sum* $S_0 = \frac{1}{2}[f(x_0+0)+f(x_0-0)]$ *at every point of discontinuity* (see Chapter IV, § 2, sec. 5).

DINI'S TEST. *The Fourier series oj a function* $f(x)$ *is convergent at a point* x_0 *to the sum* S_0 *if, given some* $h > 0$, *the integral*

$$\int_0^h \frac{|\varphi(t)|}{t}\,dt$$

exists, where

$$\varphi(t) = f(x_0+t)+f(x_0-t)-2S_0.$$

The following is a particular case of Dini's test:

LIPSCHITZ'S TEST. *The Fourier series of a function* $f(x)$ *is convergent at a point* x_0 *which it is continuous to the sum* $S_0 = f(x_0)$ *if, given sufficiently small* $t > 0$,

$$|f(x_0 \pm t)-f(x_0)| \leqq Lt_\alpha,$$

where L and α are positive constants $(\alpha \leqq 1)$.

In particular, piecewise differentiable functions are suited to this test.

The Dini and Dirichlet–Jordan tests are not consequences of one another.

In the case of a function $f(x)$, only specified on the semi-interval $(-\pi, \pi)$, or defined outside it but non-periodic, application of the above theory is made possible by replacing $f(x)$ by an auxiliary function $f^*(x)$, having the following properties:

$$f^*(x) = f(x), \qquad (-\pi < x \leqq \pi),$$
$$f^*(-\pi) = f^*(\pi),$$

while $f^*(x)$ is extended to the remaining real values of x in accordance with the law of periodicity.

All the above theorems and propositions are applicable to the function $f^*(x)$. The series represents the function $f(x)$ inside the interval.

This construction is not required if $f(-\pi) = f(\pi)$. In this case the Fourier series is convergent to $f(x)$ throughout the open interval (excluding the ends). But outside the interval, the sum of the series in general no longer coincides with $f(x)$, assuming the latter is defined throughout the real axis.

On bearing in mind the footnote at the start of the present section, we can say that a function specified in any manner in any interval can be expanded as a trigonometric series in a very wide class of cases (this includes piecewise differentiable and piecewise monotonic functions) (see also Chapter IV, § 2, sec. 5).

Expansions in sines only or cosines only. The Fourier series of an even function $f(x) = (f(-x))$ contains cosines only:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

The Fourier series of an odd function $(f(x) = -f(-x))$ contains sines only:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx.$$

Every function $f(x)$, specified in the segment $[-\pi, \pi]$, can be written as the sum of an even function $f_1(x)$ and an odd function $f_2(x)$:

$$f(x) = f_1(x) + f_2(x),$$

where the component functions are

$$f_1(x) = \frac{f(x)+f(-x)}{2}, \quad f_2(x) = \frac{f(x)-f(-x)}{2}.$$

If a function is only specified in the segment $[0, \pi]$, we can define it arbitrarily in the interval $[-\pi, 0]$ and thus obtain different Fourier series, which will have a sum S_0 at a point x_0 between 0 and π, the sum being convergent either to $f(x_0)$ or to $\frac{1}{2}[f(x_0+0)+f(x_0-0)]$ (in the case of a discontinuity).

If we complete the definition of the function in $(-\pi, 0)$ as an even function, i.e. put $f(-x) = f(x)$, we get an expansion in cosines only; and if we complete it as an odd function $(f(-x) = -f(x))$, we get an expansion in sines only.

Example 37. $f(x) = x$ $(0 \leqq x \leqq \pi)$.

(1) Let $f(x) = f(-x) = -x$ $(-\pi \leqq x < 0)$; then

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right).$$

(2) Let $f(x) = -f(-x) = x$ $(-\pi < x < 0)$; then

$$f(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right).$$

The order of the Fourier coefficients. If a periodic function $f(x)$ has a finite number of discontinuities of the first kind (in a period), its Fourier coefficients will be infinitesimals of the form $O(1/n)$ as $n \to \infty$ (see Chapter 1, § 3, sec. 8 and 15).

This means that

$$n\,|a_n| < M, \quad n\,|b_n| < M,$$

where M is a constant positive number independent of n.

If the periodic function $f(x)$ is continuous everywhere, its Fourier coefficients will be infinitesimals of higher order than $1/n$ as $n \to \infty$, i.e. of the form $o(1/n)$ (see Chapter I, § 3, sec. 8 and 15).

This means that

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} nb_n = 0.$$

If a periodic continuous function has continuous derivatives everywhere up to and including the $(m-1)$th order, its Fourier coefficients a_n and b_n will be of higher order than $1/n^m$ as $n \to \infty$, i.e.

$$\lim_{n\to\infty} n^m a_n = \lim_{n\to\infty} n^m b_n = 0.$$

In particular, if a periodic function $f(x)$ has continuous derivatives of any order everywhere, its Fourier coefficients a_n and b_n satisfy the condition

$$n^m a_n \to 0, \quad n^m b_n \to 0$$

as $n \to \infty$, independently of m.

Integration of Fourier series. Let us first consider a continuous function $f(x)$, specified everywhere on E_1.

THEOREM 15. *If an absolutely integrable function $f(x)$ is specified by its Fourier series:*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1} (a_n \cos nx + b_n \sin nx), \tag{3.133}$$

we can find $\int_a^b f(x)\,dx$ by term by term integration of series (3.133), *independently of whether the latter is convergent or not, i.e.*

$$\int_a^b f(x)\,dx = \frac{a_0}{2}(b-a) + \sum_{n=1}^{\infty} \frac{a_n(\sin nb - \sin na) - b_n(\cos nb - \cos na)}{n}. \tag{3.134}$$

THEOREM 16. *Let an absolutely integrable function be specified by its Fourier series* (3.133) *(convergent or not). The following Fourier expansion now holds for its integral:*

$$\int_0^x f(x)\,dx = \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \frac{-b_n \cos nx + (a_n + (-1)^{n+1} a_0) \sin nx}{n}$$
$$(-\pi < x < \pi). \tag{3.135}$$

A particular case of this theorem is when $a_0 = 0$ (the other conditions are retained); we now have for all x:

$$\int_0^x f(x)\,dx = \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \frac{-b_n \cos nx + a_n \sin nx}{n}. \tag{3.136}$$

Differentiation of Fourier series.

THEOREM 17. *Let $f(x)$ be a continuous function of period 2π, possessing an absolutely integrable derivative (which may not exist at a finite number of points). The Fourier series for $f'(x)$ can now be obtained from the Fourier series* (3.133) *for $f(x)$ by term by term differentiation:*

$$f'(x) \sim \sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx). \tag{3.137}$$

THEOREM 18. *Let $f(x)$ be a continuous function of period 2π, possessing m derivatives, the first $m-1$ derivatives being continuous, and the m-th absolutely integrable (the m-th m-ay cease to exist at a finite number of points). Now:* (1) *The Fourier series of all m derivatives can be obtained by term by term differentiation of the Fourier series for $f(x)$, all these series, except possibly the last, being convergent to the corresponding derivative;* (2) *the following relationships hold for the Fourier coefficients of $f(x)$ (see above):*

$$\lim_{n \to \infty} n^m a_n = \lim_{n \to \infty} n^m b_n = 0. \tag{3.138}$$

The series for $f(x)$ and all the series obtained from it by term by term differentiation (except possibly the last) are uniformly convergent in this case.

This theorem has the following converse.

THEOREM 19. *Given the trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{3.139}$$

if the following relationships hold for the coefficients a_n and b_n,

$$|n^m a_n| \leqq M, \quad |n^m b_n| \leqq M (m \geqq 2,\ M = \text{const}),$$

then the sum of the series is a continuous function of period 2π, possessing $m - 2$ continuous derivatives, which may be obtained by term by term differentiation.

THEOREM 20. *Let $f(x)$ be a continuous function specified on the segment $[-\pi, \pi]$ and having an absolutely integrable derivative (which may cease to exist at a finite number of points). Then*

$$f'(x) \sim \frac{c}{2} + \sum_{n=1}^{\infty} [(nb_n + (-1)^n c) \cos nx - na_n \sin nx], \tag{3.140}$$

where a_n and b_n are the Fourier coefficients of $f(x)$, while the constant c is given by the equation

$$c = \frac{1}{\pi}[f(\pi)-f(-\pi)]. \tag{3.141}$$

THEOREM 21. *Given the series*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{3.142}$$

if the series

$$\frac{c}{2}+\sum_{n=1}^{\infty}[(nb_n+(-1)^n c)\cos nx - na_n \sin nx], \tag{3.143}$$

where

$$c = \lim_{n\to\infty}[(-1)^{n+1} nb_n], \tag{3.144}$$

is the Fourier series of an absolutely integrable function $\varphi(x)$†, *then the series* (3.142) *is the Fourier series of $f(x) = \int_0^x \varphi(x)\,dx + \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n$, continuous for $-\pi < x < \pi$, and is convergent to this function, while obviously $f'(x) = \varphi(x)$ at all the points of continuity of $\varphi(x)$.*

THEOREM 22. *Given the series*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(-1)^n(a_n \cos nx + b_n \sin nx), \tag{3.145}$$

where a_n and b_n are positive, if na_n, nb_n do not increase (as from a certain n) and tend to zero as $n \to \infty$, the series is convergent for $-\pi < x < \pi$ and has a differentiable sum $f(x)$, where

$$f'(x) = \sum_{n=1}^{\infty}(-1)^n n(b_n \cos nx - a_n \sin nx),$$

i.e. series (3.145) *can be differentiated term by term.*

Similar theorems can be stated for functions specified on the segment $[0, \pi]$ (see for example ref. 10).

Further information on trigonometric series will be found in Chapter IV.

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† We do not assume the convergence of series (3.143).

6. Asymptotic series

If a function $F(x)$, defined for $x \geqq x_0$, is to be considered for large values of the argument x, it is sometimes useful to find a function $S(x)$ of simple structure such that

$$R(x) = F(x) - S(x) \to 0 \quad \text{as} \quad x \to \infty.$$

In this case $F(x)$ can be replaced by $S(x)$ for large x. If $F(x)$ admits of the expansion (for $x > x_0$):

$$F(x) = A_0 + \frac{A_1}{x} + \ldots + \frac{A_n}{x^n} + \frac{A_{n+1}}{x^{n+1}} + \ldots = \sum_{n=0}^{\infty} \frac{A_n}{x^n}, \tag{3.146}$$

we find, on putting

$$s_n(x) = \sum_{k=0}^{n} \frac{A_k}{x^k} = A_0 + \frac{A_1}{x} + \ldots + \frac{A_n}{x^n} \tag{3.147}$$

and

$$R_n(x) = F(x) - s_n(x) = \sum_{k=n+1}^{\infty} \frac{A_k}{x^k} = \frac{A_{n+1}}{x^{n+1}} + \ldots, \tag{3.148}$$

that

$$\lim_{n \to \infty} x^n R_n(x) = 0, \qquad \text{or} \qquad R_n(x) = o\left(\frac{1}{x^n}\right), \tag{3.149}$$

i.e. $R_n(x)$ is an infinitesimal of higher order than n.

If $F(x)$ does not admit of an expansion (3·146), it is still sometimes possible to choose a series of the form (3·146) such that condition (3·149) is satisfied for any fixed n.

Series (3·147) is said to be *asymptotic* for $F(x)$, and we write

$$F(x) \sim \sum_{k=0}^{\infty} \frac{A_k}{x^k}. \tag{3.150}$$

Series (3·147) can be divergent, but it is still useful since it yields the approximate formulae

$$F(x) \approx A_0 + \frac{A_1}{x} + \ldots + \frac{A_n}{x^n},$$

the degree of the approximation being indicated by (3·149).

Example 38. Let us consider the function

$$F(x) = \int_x^{\infty} e^{x-t} \frac{dt}{t}.$$

Repeated integration by parts gives:

$$F(x) = \frac{1}{x}-\frac{1}{x^2}+\frac{2!}{x^3}- \dots +(-1)^{n-1}\frac{(n-1)!}{x^n}+R_n(x),$$

where

$$R_n(x) = (-1)^n n! \int_x^\infty \frac{e^{x-t}}{t^{n+1}}\,dt,$$

$$\int_x^\infty \frac{e^{x-t}}{t^{n+1}}\,dt = -\frac{e^{x-t}}{t^{n+1}}\Bigg|_{t=x}^{t=\infty} -(n+1)\int_x^\infty \frac{e^{x-t}}{t^{n+2}}\,dt < \frac{1}{x^{n+1}},$$

consequently,

$$|R_n(x)| < \frac{n!}{x^{n+1}}$$

and condition (3.149) is satisfied. Thus

$$F(x) \sim \frac{1}{x}-\frac{1}{x^2}+\frac{2!}{x^3}- \dots +(-1)^{n-1}\frac{(n-1)!}{x^n}+ \dots .$$

This series is divergent, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{(n-1)!}{x^n} = \infty.$$

If $F(x)$ admits of an *asymptotic* expansion, this latter is unique.

The asymptotic expansion of the product of two functions $F(x)$ and $G(x)$, each of which has an asymptotic expansion

$$F(x) \sim \sum_{k=0}^{\infty} A_k x^{-k}, \quad G(x) \sim \sum_{k=0}^{\infty} B_k x^{-k}, \tag{3.151}$$

is formally equal to the product of asymptotic expansions (3.151):

$$F(x)G(x) \sim \sum_{k=0}^{\infty} A_k x^{-k} \sum_{k=0}^{\infty} B_k x^{-k} = \sum_{k=0}^{\infty} C_k x^{-k}, \tag{3.152}$$

where

$$C_m = \sum_{k=0}^{m} A_k B_{m-k}. \tag{3.153}$$

If the asymptotic expansion (3.150) of $F(x)$ starts with the term A_2x^{-2}, it can be *formally integrated term by term* in the interval from x to $+\infty$, i.e.

$$\int_x^\infty F(x)\,dx \sim \sum_{k=2}^{\infty}\int_x^\infty \frac{A_k}{x^k}\,dx = \sum_{k=2}^{\infty}\frac{A_k}{(k-1)x^{k-1}}.$$

Formal differentiation of asymptotic expansion is not permissible.

If $F(x)$ has an asymptotic expansion with no free term ($A_0 = 0$ and $F(x) \to 0$ as $x \to \infty$), this expansion can be exponentiated

$$e^{F(x)} = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} [F(x)]^m \sim$$

$$\sim 1 + \frac{A_1}{x} + \ldots + \left[\frac{A_1^n}{n!} + \ldots + \frac{A_n}{1!}\right] \frac{1}{x^n} + \ldots . \quad (3.154)$$

The asymptotic expansion is thus a source of approximate formulae for computing a function for large values of its argument, the accuracy of formula (3.147) being the greater, the greater the value of the argument.

There exist functions, not identically zero, for which all the coefficients A_k in the asymptotic form (3·150) are equal to zero. Such functions are described as asymptotic zeros. An *asymptotic zero* is any function $F(x)$, for which

$$F(x) = O\left(\frac{1}{x^n}\right)$$

for any n. For example, $F(x) = e^{-x}$ is an asymptotic zero. Addition of this function to the left-hand side of (3·150) does not change its right-hand side.

Not every function $F(x)$, defined in the semi-interval $[a, +\infty]$, admits of an asymptotic expansion (3·150), but if such an expansion exists, it is unique for the given $F(x)$ (the coefficients are uniquely defined). On the other hand, there exists for any sequence of numbers $\{A_k\}$ a function $F(x)$, for which the A_k are the coefficients of an asymptotic expansion (3·150). This $F(x)$ is not uniquely defined, however (to an accuracy of an asymptotic zero).

7. Some methods of generalized summation of divergent series

Certain classes of divergent series, i.e. series lacking a sum in the ordinary sense of the word, are capable of *generalized summation.*

The definition of generalized sum usually satisfies two conditions:

1°. *The linearity condition.* If the generalized sum A corresponds to the series $\sum a_n$, and the generalized sum B to $\sum b_n$, the series

$\sum(pa_n+qb_n)$, where p and q are arbitrary constants, must have a generalized sum equal to $pA+qB$.

2°. *The permanence condition.* If a series is convergent in the ordinary sense to the sum A, it must also have a generalized sum, equal to A.

The general plan for constructing linear permanent methods of summation is as follows.

Let

$$\gamma_0(x),\ \gamma_1(x),\ \dots,\ \gamma_n(x),\ \dots \tag{3.155}$$

be a sequence of functions given in a dòmain X of variation of the parameter x, and let X have a *point of condensation* — the finite or improper number ω.

We construct, in accordance with the numerical series

$$\sum_{n=0}^{\infty} a_n, \tag{3.156}$$

the functional series

$$\sum_{n=0}^{\infty} a_n\gamma_n(x). \tag{3.157}$$

If this series is convergent, at least for x sufficiently close to ω, and its sum $S(x)$ tends to a limit A as $x \to \omega$, this number A is in fact taken as the *generalized sum* of series (3.156). To ensure that the method has permanence, two conditions are imposed on the functions $\gamma_n(x)$:

(1)
$$\lim_{x\to\omega} \gamma_n(x) = 1 \qquad (n = 0,\ 1,\ 2,\ \dots); \tag{3.158}$$

(2) for all $x \in X$:

$$|\gamma_0(x)| + \sum_{n=1}^{\infty} |\gamma_n(x)-\gamma_{n-1}(x)| \leqq K < \infty \qquad (K = \text{const}). \tag{3.159}$$

We shall next consider two methods of generalized summation, embraced by the general plan just described.

1°. *The method of power series* (Poisson). In accordance with the given numerical series (3.156), we form the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + \dots + a_nx^n + \dots; \tag{3.160}$$

if this series is convergent in the interval $0 < x < 1$ and its sum $S(x)$ has the limit A as $x \to 1$:

$$\lim_{n \to 1-0} S(x) = A,$$

the number A is called the *generalized sum (in Poisson's sense)* of the given series (3.156).

This method follows from the general scheme if we put

$$X = (0, 1), \quad \omega = 1, \quad \gamma_n(x) = x^n \qquad (n = 0, 1, 2, \ldots).$$

EXAMPLE 39. Let us take the series discussed by Euler:

$$1-1+1-1+1-1+\ldots. \tag{3.161}$$

It has no sum in the ordinary sense, since s_n oscillates between 0 and +1. The corresponding power series (3.160) has the form

$$1-x+x^2-x^3+x^4-x^5+\ldots = \sum_{n=0}^{\infty}(-1)^n x^n.$$

For $0 < x < 1$, its sum (as the sum of an infinite geometrical progression) is equal to

$$S(x) = \frac{1}{1+x}; \tag{3.162}$$

the generalized sum in Poisson's sense of series (3.161) is

$$A = \lim_{x \to 1-0} \frac{1}{1+x} = \frac{1}{2}. \tag{3.163}$$

EXAMPLE 40. The series

$$\sum_{n=1}^{\infty} \sin n\theta \qquad (-\pi \leqq \theta \leqq \pi) \tag{3.164}$$

is convergent only for $\theta = 0$ and $\theta = \pm\pi$. The corresponding power series

$$\sum_{n=1}^{\infty} x^n \sin n\theta \tag{3.165}$$

has the sum, for $0 < x < 1$:

$$S(x) = \frac{x \sin\theta}{1-2x\cos\theta + x^2}.$$

The generalized sum is thus equal to

$$A=\lim_{x\to 1-0}\frac{x\sin\theta}{1-2x\cos\theta+x^2}=\frac{\sin\theta}{2(1-\cos\theta)}=\frac{1}{2}\operatorname{cotan}\frac{\theta}{2}. \tag{3.166}$$

2°. *The method of arithmetic means* (Cesàro). We take the partial sums s_n of the given numerical series (3.156):

$$\left.\begin{aligned} s_0&=a_0,\quad s_1=a_0+a_1,\\ s_2&=a_0+a_1+a_2,\ \ldots,\quad s_n=\sum_{k=0}^{n}a_k \end{aligned}\right\} \tag{3.167}$$

and form their consecutive arithmetic means:

$$A_1=s_0,\quad A_2=\frac{s_0+s_1}{2},\ \ldots,\ A_n=\frac{s_0+s_1+\ldots+s_{n-1}}{n}. \tag{3.168}$$

If the sequence $\{A_n\}$ has a limit A as $n\to\infty$, we call A the *generalized sum (in Cesàro's sense)* of series (3.156).

This method follows from the general plan if we put

$$X=n,\ \omega=+\infty,\ \gamma_m(n)=\begin{cases}1-\dfrac{m}{n} & \text{for}\quad m=0,1,\ldots,n-1,\\ 0 & \text{for}\quad m\geqq n.\end{cases}$$

Example 41. Let us return to series (3.164). When $\varphi\neq 0$:

$$\left.\begin{aligned} s_n&=\frac{\cos\frac{1}{2}\theta-\cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta},\\ nA_n&=\frac{n}{2}\operatorname{cotan}\frac{1}{2}\theta-\frac{\sin(n+1)\theta-\sin\theta}{4\sin^2\frac{1}{2}\theta},\\ A&=\lim_{n\to\infty}A_n=\frac{1}{2}\operatorname{cotan}\frac{\theta}{2}. \end{aligned}\right\}$$

We have thus obtained a generalized sum in Cesàro's sense equal to the generalized sum in Poisson's sense.

The inter-relationship between the Cesàro and Poisson methods may be explained by quoting the following theorem:

THEOREM 23. (Frobenius). *If a series is summable in accordance with the method of arithmetic means to the finite "sum" A, it is simultaneously summable by the method of power series, to the same sum A.*

Notice that Poisson's method is applicable to a wider class of series than Cesàro's method, though it does not contradict Cesàro's method when both are applicable.

§ 3. Methods of calculating the sum of a series

We shall discuss in this section some methods of finite summation of series, some estimates for series, finite sums and products, and the problem, of importance for practical computation, of improving the convergence of series, i.e. methods enabling a given series to be replaced by another, having the same sum but more rapidly convergent.

1. Elementary methods of exact summation

It is very rarely possible to sum exactly an infinite series obtained as a result of solving some problem. Some simple methods of strict finite summation are given below.

THEOREM 24. *If the terms a_n of the series $\sum\limits_{n=0}^{\infty} a_n$ can be written in the form $a_n = b_n - b_{n+1}$ and if the b_n form a sequence $\{b_n\}$ having a limit a, the sum of the series is*

$$S = \sum_{n=0}^{\infty} a_n = b_0 - a. \tag{3.169}$$

EXAMPLE 42. Suppose we have the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}\left(\sqrt{n}+\sqrt{n+1}\right)};$$

here

$$\frac{1}{\sqrt{n(n+1)}\left(\sqrt{n}+\sqrt{n+1}\right)} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}},$$

and $\lim\limits_{n\to\infty} 1/\sqrt{n} = 0$, so that $S = 1$ $(b_0 = b_1 = 1)$.

THEOREM 25. *If the terms of the series* $\sum\limits_{n=0}^{\infty} a_n$ *are expressible as*

$$a_n = \alpha_1 b_{n+1} + \alpha_2 b_{n+2} + \ldots + \alpha_p b_{n+p}, \tag{3.170}$$

where p is some fixed positive integer $\geqq 2$, b_n *form a sequence having a limit a, and* α_i *are numbers satisfying the condition*

$$\alpha_1 + \alpha_2 + \ldots + \alpha_p = 0, \tag{3.171}$$

the series in question is convergent, and its sum is equal to

$$S = \alpha_1 b_1 + (\alpha_1 + \alpha_2) b_2 + \ldots + (\alpha_1 + \alpha_2 + \ldots + \alpha_{p-1}) b_{p-1} + \\ + (\alpha_2 + 2\alpha_3 + 3\alpha_4 + (p-1)\alpha_p) a. \tag{3.172}$$

EXAMPLE 43. Given the series

$$\sum_{n=0}^{\infty} \frac{4n+6}{(n+1)(n+2)(n+3)},$$

we have

$$\frac{4n+6}{(n+1)(n+2)(n+3)} = \frac{1}{n+1} + \frac{2}{n+2} - \frac{3}{n+3}.$$

Consequently, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = -3$.

Condition (3.171) is satisfied : $1+2-3 = 0$. In addition, $b_n = 1/n \to 0$ as $n \to \infty$. We now find, by (3.172):

$$S = 1.1 + (1+2)\frac{1}{2} = \frac{5}{2}.$$

THEOREM 26. *If the terms of the series* $\sum\limits_{n=0}^{\infty} a_n$ *can be written as*

$$a_n = b_n - b_{n+q},$$

where q is a positive integer, while the sequence of numbers b_0, b_1, $b_2, \ldots$ *has a limit* ξ, *the sum of the series is*

$$S = \sum_{n=0}^{\infty} a_n = b_0 + b_1 + \ldots + b_{q-1} - q\xi.$$

Tables of series expansions of functions and the relevant sections of reference works may be used for the finite summation of series. Given a particular series, it should be verified whether or not it is to be found in a reference book, or whether it can be transformed to a familiar series by a change of variable or some other means.

EXAMPLE 44. Suppose we have the series

$$\sum_{k=1}^{\infty} kp^k \sin kx, \quad |p| < 1.$$

We find in ref. 7, formula 1.447 3, which we can write as

$$\sum_{k=1} p^k \cos kx = \frac{1}{2}\,\frac{1-p^2}{1-2p\cos x+p^2}-\frac{1}{2} \qquad (|p| < 1).$$

The uniform convergence of this series and the series of the derivatives of its terms with respect to x follows from comparing them with the majorant series $\sum_{k=1}^{\infty} |p|^k$ and $\sum_{k=1}^{\infty} k\,|p|^k$ respectively. On differentiating both sides of the equation with respect to x, we find the sum of the series (see § 2, sec. 1):

$$\sum_{k=1}^{\infty} kp^k \sin kx = \frac{p(1-p^2)\sin x}{(1-2p\cos x+p^2)^2} \qquad (|p| < 1).$$

2. Summation of series with the aid of functions of a complex variable

If the trigonometric series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx = f_1(x), \tag{3.173}$$

$$\sum_{n=1}^{\infty} a_n \sin nx = f_2(x) \tag{3.174}$$

have positive coefficients a_n, and the series $\sum_{n=1}^{\infty} a_n/n$ is convergent, (3.173) and (3.174) are the Fourier series of continuous functions.

We can sometimes find the functions $f_1(x)$ and $f_2(x)$ — the sums of series (3.173) and (3.174) — by making use of functions of a complex variable.

Let

$$\varphi(z) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n z^n$$

be the sum of a series convergent in the circle $|z| < 1$. If $\lim_{|z|\to 1} \varphi(z) =$

$= \lim\limits_{r\to 1} \varphi(re^{ix}) = \varphi(e^{ix})$, then $f_1(x)+if_2(x) = \varphi(e^{ix})$, i.e. after separating the real and imaginary parts of $\varphi(e^{ix})$, we have:

$$f_1(x) = \operatorname{Re}\{\varphi(e^{ix})\}; \tag{3.175}$$

$$f_2(x) = \operatorname{Im}\{\varphi(e^{ix})\}. \tag{3.176}$$

EXAMPLE 45. Let us establish the convergence and sum of the series

$$1+\frac{\cos x}{1!}+\frac{\cos 2x}{2!}+\ldots = 1+\sum_{n=1}^{\infty}\frac{\cos nx}{n!}, \tag{3.177}$$

$$\frac{\sin x}{1!}+\frac{\sin 2x}{2!}+\ldots = \sum_{n=1}^{\infty}\frac{\sin nx}{n!}. \tag{3.178}$$

The convergence of series (3.117) and (3.178) follows readily from a comparison with the series

$$1+\sum_{n=1}^{\infty}\frac{1}{n!} = e,$$

which is majorant for the given series. Furthermore, we put

$$\varphi(z) = 1+\sum_{n=1}^{\infty}\frac{z^n}{n!} = e^z.$$

On expressing z in the exponential form $z = re^{ix}$ and letting r tend to 1, we have

$$e^{e^{ix}} = e^{\cos x+i\sin x} = e^{\cos x}[\cos(\sin x)+i\sin(\sin x)] =$$

$$= 1+\sum_{n=1}^{\infty}\frac{(\cos x+i\sin x)^n}{n!},$$

$$e^{\cos x}\cos(\sin x)+ie^{\cos x}\sin(\sin x) = 1+\sum_{n=1}^{\infty}\frac{\cos nx}{n!}+i\sum_{n=1}^{\infty}\frac{\sin nx}{n!}.$$

Hence,

$$1+\sum_{n=1}^{\infty}\frac{\cos nx}{n!} = e^{\cos x}\cos(\sin x), \tag{3.179}$$

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n!} = e^{\cos x}\sin(\sin x). \tag{3.180}$$

3. Summation of series with the aid of Laplace transforms

Definition of the Laplace transform. Suppose that, given any $x > 0$, the modulus of the function $\varphi(x)$ increases more slowly than some exponential function i.e. there exist numbers M and s_0, not depending on x, such that, for any x,

$$|\varphi(x)| < Me^{s_0 x}. \tag{3.181}$$

Let $p = s+i\sigma$ be a complex number. The integral

$$a(p) = \int_0^\infty e^{-px}\varphi(x)\,dx \tag{3.182}$$

now exists and has derivatives of all orders in the half-plane $\operatorname{Re} p > s_0$. The integral (3.182) is called the *Laplace transform* of $\varphi(x)$. In the accepted brief terminology, $\varphi(x)$ is the *pre-image*, and $a(p)$ the *image*.

Tables have been compiled for the pre-images and images of many functions (see for example ref. 7).

Summation of numerical series (ref. 9). If, given the convergent infinite series

$$S = \sum_{k=1}^\infty (\pm 1)^k a(k) \tag{3.183}$$

its terms are images of a pre-image $\varphi(\xi)$, i.e.

$$a(k) = \int_0^\infty e^{-k\varepsilon}\varphi(\varepsilon)\,d\varepsilon,$$

the following formula holds (preserving the correspondence of the $\pm$ signs):

$$\sum_{k=1}^\infty (\pm 1)^k a(k) = \pm\int_0^\infty \frac{\varphi(\xi)\,d\xi}{e\xi \mp 1}. \tag{3.184}$$

Example 46. Knowing that

$$\frac{a}{k^2+a^2} = \int_0^\infty e^{-k\xi}\sin a\xi\,d\xi,$$

we have by (3.184):

$$\sum_{k=1}^\infty \frac{1}{k^2+a^2} = \frac{1}{a}\int_0^\infty \frac{\sin a\xi\,d\xi}{e\xi-1} = \frac{\pi}{2a}\operatorname{cotanh}\pi a - \frac{\pi}{a^2}.$$

Generating functions. If the terms of the sequence $\{f_k(t)\}$ ($k = 1, 2, 3, \dots$), defined in some interval $\alpha < t < \beta$, are the coefficients or the Taylor expansion of a known function $F(x, t)$ in powers of x fof $|x| < 1$, i.e.

$$F(x, t) = \sum_{k=1}^{\infty} x^k f_k(t) \tag{3.185}$$

$F(x,t)$ is called the *generating function* of the sequence $\{f_k(t)\}$. For example, one of the following sequences can be taken as $\{f_k(t)\}$:

(a) the sequence of powers $t, t^2, t^3, \dots$, where

$$F(x, t) = \frac{xt}{1-xt} = \sum_{k=1}^{\infty} x^k t^k. \tag{3.186}$$

and $|xt| < 1$;

(b) the sequence of trigonometric functions

$$\sin t, \quad \sin 2t, \quad \sin 3t, \dots$$

or

$$\cos t, \quad \cos 2t, \quad \cos 3t, \dots$$

where

$$F_1(x, t) = \frac{x \sin t}{1-2x\cos t + x^2} = \sum_{k=1}^{\infty} x^k \sin kt \tag{3.187}$$

and

$$F_2(x, t) = \frac{x\cos t - x^2}{1-2x\cos t+x^2} = \sum_{k=1}^{\infty} x^k \cos kt. \tag{3.188}$$

where t can vary in any interval and $|x| < 1$;

(c) the sequence of *Chebyshev polynomials* $T_1(x), T_2(x), T_3(x), \dots$, for which the following relationship holds:

$$F(x, t) = \frac{x(t-x)}{1-2xt+x^2} = \sum_{k=1}^{\infty} x^k T_k(t). \tag{3.189}$$

where $|t| \leqq 1, |x| < 1$;

(d) the sequence of *Legendre polynomials* $P_1(t), P_2(t), P_3(t), \dots$, where

$$F(x, t) = \frac{1}{\sqrt{x^2-2xt+1}} - 1 = \sum_{k=1}^{\infty} x^k P_k(t), \tag{3.190}$$

for $|t| \leqq 1$ and $|x| < 1$.

It should also be noticed that the infinite series (3.186)-(3.190) are convergent for $x = \pm 1$ or oscillate between finite limits (i.e. the sum of the first m terms remains less in absolute value than some constant independent of t and m).

The summation of functional series (ref. 9). Suppose that the generating function $F(x, t)$ is known for an infinite sequence of functions $\{f_k(t)\}$ $(k = 1, 2, 3, \ldots)$ (see above). Let the series

$$S(t, x) = \sum_{k=1}^{\infty} a(k) x^k f_k(t) \tag{3.191}$$

be convergent in some interval $\alpha < t < \beta$ and for $|x| \leqq 1$. If the coefficients $a(k)$ of series (3.191), considered as functions of the index k, are images of some $\varphi(\xi)$, we have for the sum of series (3.191):

$$S(t, x) = \int_0^{\infty} \varphi(\xi) F(xe^{-\xi}, t)\, d\xi, \tag{3.192}$$

Example 47. Knowing that the fraction $1/(k+l)$ is an image of $e^{-\xi l}$ for any $l \geqq 0$ and $k > 0$, i.e.

$$\frac{1}{k+l} = \int_0^{\infty} e^{-(k+l)\xi}\, d\xi,$$

and using formulae (3.187), (3.188) and (3.192) for positive integral l, we have:

$$\begin{aligned} S_1(t, x) &= \sum_{k=1}^{\infty} \frac{x^k \sin kt}{k+l} = \\ &= x^{-l} \left\{ \sum_{m=1}^{l} \frac{c_l^m (-1)^{m-1}}{m} \left[\sum_{r=0}^{m} C_m^r (-1)^r x^r \sin (r-l)t + \sin lt \right] - \right. \\ &\left. - \frac{\sin lt}{2} \ln (1 - 2x \cos t + x^2) + \cos lt \arctan \frac{x \sin t}{1 - x \cos t} \right\}; \end{aligned} \tag{3.193}$$

$$\begin{aligned} S_2(t, x) &= \sum_{k=l}^{\infty} \frac{x^k \cos kt}{k+l} = \\ &= x^{-l} \left\{ \sum_{m=1}^{l} \frac{c_l^m (-1)^{m-1}}{m} \left[\sum_{r=0}^{m} C_m^r (-1)^r x^r \cos (r-l)t - \cos lt \right] - \right. \\ &\left. - \frac{\cos lt}{2} \ln (1 - 2x \cos t + x^2) + \sin lt \arctan \frac{x \sin t}{1 - x \cos t} \right\}. \end{aligned} \tag{3.194}$$

Formulae (3·193) and (3·194) remain valid for any t and $|x| \leqq 1$, with the exception of points of discontinuity (ref. 11), which may make their appearance for the trigonometric series with $x = \pm 1$.

On making use of generating functions (3·186), (3·189), (3·190), as also (3·192) with positive integral l, summation formulae similar to 3·193) and (3·194) can be established for the series

$$\sum_{k=1}^{\infty} \frac{t^k}{k+l}, \quad \sum_{k=1}^{\infty} \frac{x^k T_k(t)}{k+l} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{x^k p_k(t)}{k+l}$$

4. Integral estimations for finite sums and infinite series

Euler's formula. If a function $a(x)$ has derivatives up to and including the nth order for $m \leqq x \leqq k$ (m and k are positive integers), the following formula holds, due to Euler:

$$\sum_{\tau=m}^{k-1} a(\tau) = \int_m^k a(t)\,dt + \sum_{\nu=1}^{n-1} \frac{B_\nu}{\nu!}[a^{(\nu-1)}(k) - a^{(\nu-1)}(m)] - \\ - \frac{1}{n!}\int_0^1 Y_n(t) \sum_{\tau=m}^{k-1} a^{(n)}(\tau+1-t)\,dt^{\dagger}, \qquad (3.195)$$

where $Y_n(t) = B_n(t) - B_n$, B_n are *Bernoulli numbers*, $B_n(t)$ are *Bernoulli polynomials* (see Chapter VI).

First integral estimate. We suppose that $a^{(n)}(x) \geqq 0$[††] in the interval $q \leqq x \leqq p+1$ (p and q are positive integers). Let M_n denote the greatest, and m_n the least, value of $Y_n(t)$ in the segment $0 \leqq t \leqq 1$. The following inequality now holds:

$$s_n(p+1,\ q) - \frac{M_n}{n!}[a^{(n-1)}(p+1) - a^{(n-1)}(q)] \leqq \sum_{k=q}^{p} a(k) \leqq \\ \leqq s_n(p+1,\ q) - \frac{m_n}{n!}[a^{(n-1)}(p+1) - a^{(n-1)}(q)], \qquad (3.196)$$

[†] When $n=1$ we have to assume $\sum_{\nu=1}^{n-1} (B_\nu/\nu!)\,[a^{(\nu-1)}(k) - a^{(\nu-1)}(m)] = 0$; in addition, it is assumed here and throughout what follows that $f^{(0)}(x) =$ [illegible]

[††] If $a^{(n)}(x) \leqq 0$, the inequality signs of (3.196) have to be reversed.

where[†]

$$s_n(p+1,\ q) = \int_q^{p+1} a(t)\,dt + \sum_{\nu=1}^{n-1} \frac{B_\nu}{\nu!}[a^{(\nu-1)}(p+1) - a^{(\nu-1)}(q)].$$

Let us quote some values for M_n and m_n:

$$M_1 = 1,\ m_1 = 0;\ M_2 = 0,\ m_2 = \frac{1}{4};\ M_3 = \frac{\sqrt{3}}{36},\quad m_3 = \frac{\sqrt{3}}{36};$$

$$M_4 = \frac{1}{16},\ m_4 = 0;\ M_5 \approx 0{\cdot}0244582\ldots,\ m_5 \approx -0{\cdot}0244582\ldots.$$

It can be shown that

$$M_{2l} = Y_{2l}\left(\frac{1}{2}\right) > 0 \quad \text{and} \quad m_{2l} = 0, \quad \text{if} \quad l \text{ is even}$$

$$M_{2l} = 0 \quad \text{and} \quad m_{2l} = Y_{2l}\left(\frac{1}{2}\right) < 0, \quad \text{if} \quad l \text{ is odd}.$$

If we make the following assumptions:

(a) $a^{(n)}(x) \geqq 0$ for $q \leqq x < \infty$;

(b) $\lim\limits_{x\to\infty} a^{(m)}(x) = 0$ for $m = 0, 1, 2, \ldots, n-1$;

(c) the integral $\int_q^\infty a(x)\,dx$ is convergent,

we now get, by (3.196), as $p \to \infty$:

$$s_n(+\infty, q) + \frac{M_n}{n!} a^{(n-1)}(q) \leqq$$

$$\leqq \sum_{k=q}^{\infty} a(k) \leqq s_n(+\infty,\ q) + \frac{m_n}{n!} a^{(n-1)}(q), \tag{3.197}$$

where

$$s_n(+\infty,\ q) = \int_q^\infty a(t)\,dt - \sum_{\nu=1}^{n-1} \frac{B_\nu}{\nu} a^{(r-1)}(q).$$

EXAMPLE 48. Knowing that the derivatives of any order of $a(x) = 1/\sqrt{x}$ are sign-definite in $1 \leqq x \leqq 101$, we can estimate the sum

$$S = \sum_{k=1}^{100} \frac{1}{\sqrt{k}},$$

by making use of (3.196).

[†] When $n=1$, we have to assume $\sum_{\nu=1}^{n-1} \frac{B_\nu}{\nu!}[a^{(\nu-1)}(p+1) - a^{(\nu-1)}(q)] = 0$.

We have, for instance,

$$\begin{array}{lll} 18{\cdot}1 < S < 19 & \text{for} & n = 1; \\ 18{\cdot}55 < S < 18{\cdot}61 & \text{for} & n = 2; \\ 18{\cdot}5890 < S < 18{\cdot}5909 & \text{for} & n = 6. \end{array}$$

Second integral estimate. If we assume that $(-1)^{m-1}\,a^{(2m)}(x) \geqq 0$[†] for $q \leqq x \leqq p+1$, the following inequality holds:

$$s_{2m-1}(p+1,\ q)+\frac{B_{2m}}{(2m)!}\,[a^{(2m-1)}(p+1)-a^{(2m-1)}(q)] \geqq \sum_{k=q}^{p} a(k) \geqq s_{2m-1}(p+1,\ q), \qquad (3.198)$$

where

$$s_{2m-1}(p+1,\ q) = \int_q^{p+1} a(t)\,dt + \sum_{\nu=1}^{2m-2} \frac{B_\nu}{\nu!}\,[a^{(\nu-1)}(p+1)-a^{(\nu-1)}(q)].$$

If we assume:

(a) $(-1)^{m-1}a^{(2m)}(x) \geqq 0$[†] for $q \leqq x < +\infty$

(b) $\lim\limits_{x\to\infty} a^{(2k-1)}(x) = 0$ for $k = 1, 2, \ldots, m;$

(c) integral $\int_q^\infty a(x)\,dx$ is convergent,

the following holds:

$$s_{2m-1}(+\infty,\ q)-\frac{B_{2m}}{(2m)!}\,a^{(2m-1)}(q) \geqq \sum_{k=q}^{\infty} a(k) \geqq s_{2m-1}(+\infty,\ q), \qquad (3.199)$$

where

$$s_{2m-1}(+\infty\ q) = \int_q^\infty a(t)\,dt - \sum_{\nu=1}^{2m-2} \frac{B_\nu}{\nu!}\,a^{(\nu-1)}(q)$$ [††].

Example 49. Knowing that, with $a(x) = x^{3/2}$, the following condition holds: $(-1)^4\,a^{\mathrm{IV}}(x) > 0$ for all $x > 0$, we can carry out an estimate of the sum

$$S = \sum_{k=1}^{100} k^{3/2},$$

[†] If $(-1)^{m-1}a^{(2m)}(x) \leqq 0$, the signs of inequalities (3.198) and (3.199) must be reversed.

[††] When $m=1$, we have to put $\sum\limits_{\nu=1}^{2m-2}(B_\nu/\nu!)a^{(\nu-1)}(q) = B_1a(q)$.

by making use of inequality (3.198) with $m = 2$. We have

$$40501{\cdot}2260 < S < 40501{\cdot}2265.$$

The first and second integral estimates may be used not only for evaluating finite sums and infinite series, but also for evaluating finite and infinite products.

EXAMPLE 50. Let us estimate the size of the product $P = p!$.

On taking logarithms, we have $S = \ln P = \sum\limits_{k=1}^{p} \ln k$.

Notice that the following conditions are satisfied for the function $a(x) = \ln x$ in the interval $1 \leqq x \leqq p+1$:

$$a'(x) = \frac{1}{x} > 0, \quad a''(x) = -\frac{1}{x^2} < 0 \quad \text{and} \quad a'''(x) = \frac{2}{x^3} > 0.$$

On using (3.196), say with $n = 3$, we get

$$s_3(p+1,\ 1) - \frac{\sqrt{3}}{216}[a''(p+1) - a''(1)] \leqq \ln P \leqq$$
$$\leqq s_3(p+1,\ 1) + \frac{\sqrt{3}}{216}[a''(p+1) - a''(1)], \tag{3.200}$$

where

$$s_3'(p+1,\ 1) = \left(p+\frac{1}{2}\right)\ln(p+1) - p\left[1 - \frac{1}{12(p+1)}\right]$$

On exponentiating inequality (3.200), we finally get

$$L_p e^{-\frac{\sqrt{3}}{216}\frac{(2+p)p}{(p+1)^2}} \leqq p! \leqq L_p e^{\frac{\sqrt{3}}{216}\frac{(2+p)p}{(p+1)^2}},$$

where

$$L_p = (p+1)^{p+\frac{1}{2}} e^{-p\left[1+\frac{1}{12(p+1)}\right]}$$

We next consider transformation of series for improving their convergence.

5. Kummer's transformation

Suppose we have the series with positive terms:

$$R_m = \sum_{n=m}^{\infty} a_n \tag{3.201}$$

and some auxiliary series

$$B_m = \sum_{n=m}^{\infty} b_n \tag{3.202}$$

which is convergent and has a finite sum B_m. Suppose the finite limit exists:

$$A = \lim_{n\to\infty} \frac{b_n}{a_n} \neq 0. \tag{3.203}$$

In these circumstances series (3.201) is convergent (see Test III, § 1) and the following identity holds:

$$R_m = \frac{B_m}{A} + \sum_{n+m}^{\infty} \left(1 - \frac{1}{A}\frac{b_n}{a_n}\right) a_n, \tag{3.204}$$

which is known as *Kummer's transformation* for the series (3.201).

6. Improvement of the convergence of series corresponding to a given convergence test

Transformation (3.204) can be used for improving the convergence of series (3.201). For, by (3.204), the convergence of (3.201) will be improved in the sense that the general term $a_n^{(1)} = \left(1 - \frac{1}{A}(b_n/a_n)\right)a_n$ of the transformed series $\sum\limits_{n=m}^{\infty} a_n^{(1)}$ will tend to zero faster than a_n, i.e. $\lim\limits_{n\to\infty} a_n^{(1)}/a_n = 0$. Obviously, the faster the ratio b_n/a_n tends to its finite limit $A \neq 0$, the faster the series on the right-hand side of (3.204) will converge.

Transformation (3.204) can be applied several times in succession to the same series (3.201).

Suppose, for example, that the terms of the auxiliary series

$$B_m^{(k)} = \sum_{n=m}^{\infty} b_n^{(k)} \qquad (k = 0, 1, 2, \ldots, p)$$

are chosen successively so that

(1) the sum $B_m^{(k)}$ $(k = 1, 2, \ldots, p)$ is already known to us;

(2) $A_k = \lim\limits_{n\to\infty} A_n^{(k)}$, where $A_n^{(k)} = \dfrac{b_n^{(k)}}{a_n^{(k)}}$, $a_n^{(k+1)} = \left(1 - \dfrac{A_n^{(k)}}{A_k}\right) a_n^{(k)}$.

all the A_k being bounded and non-zero.

Now, on putting $A_n^{(k)}/A_k = 1-\varepsilon_n^{(k)}$, where $\lim\limits_{n\to\infty} \varepsilon_n^{(k)} = 0$, we have after the pth transformation of (3.201):

$$R_m = \sum_{k=0}^{p} \frac{B_m^{(k)}}{A_k} + \sum_{n=m}^{\infty} a_n \prod_{k=0}^{p} \varepsilon_n^{(k)}. \tag{3.205}$$

It occasionally turns out that all the $\varepsilon_n^{(k)} = 0$ for a certain k, and we get the strict sum of series (3.201).

Every transformation of type (3.204) is determined by the choice of some series $\sum\limits_{n=1}^{\infty} b_n$, which corresponds to some test which is sufficient for the convergence of the series $\sum\limits_{n=1}^{\infty} a_n$, obtained from Test III. We can say in this sense that, for every convergence test following from Test III, there is a corresponding method of improving the convergence of the series. It may prove convenient in practice to take all the $b_n^{(0)}$, $b_n^{(1)}$, $b_n^{(2)}$, ... equal or different in transformations (3.205), i.e. to choose the stages in the improvement of the convergence in accordance with the convergence obtained by using different tests. We shall consider below, as an example of the application of transformation (3.205), methods of improving the convergence of series corresponding to d'Alembert's and Gauss's tests.

Improvement of the convergence of series corresponding to d'Alembert's test. We put in identity (3.204): $b_n = a_{n+1}-a_n = \Delta a_n$ and take the series $\sum\limits_{n=1}^{\infty} a_n$ to be convergent by d'Alembert's test, i.e

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \varrho < 1.$$

Let $\varrho \neq 0$; then

$$R_m = \sum_{n=m}^{\infty} a_n = -\frac{a_m}{A_0} + \sum_{n=m}^{\infty} \left(a_n - \frac{\Delta a_n}{A_0}\right), \tag{3.206}$$

where

$$A_0 = \lim_{n\to\infty} \frac{\Delta a_n}{a_n}.$$

On again applying transformation (3.206) to the series

$$\sum_{n=m}^{\infty} a_n^{(1)}, \quad \text{where} \quad a_n^{(1)} = a_n - \frac{\Delta a_n}{A_0},$$

we have

$$\sum_{n=m}^{\infty} a_n^{(1)} = -\frac{1}{A_1}\left(1-\frac{\Delta}{A_0}\right)a_m + \sum_{n=m}^{\infty}\left(1-\frac{\Delta}{A_0}\right)\left(1-\frac{\Delta}{A_1}\right)a_n.$$

The following notation is used here:

$$a_n - \left(\frac{1}{A_0}+\frac{1}{A_1}\right)\Delta a_n + \frac{\Delta^2 a_n}{A_0 A_1} = \left(1-\frac{\Delta}{A_0}\right)\left(1-\frac{\Delta}{A_1}\right)a_n,$$

$$A_1 = \lim_{n\to\infty}\frac{\Delta a_n^{(1)}}{a_n^{(1)}} = \lim_{n\to\infty}\frac{\Delta\left(1-\frac{\Delta}{A_0}\right)a_n}{\left(1-\frac{\Delta}{A_0}\right)a_n}.$$

Thus, p successive applications of the above transformation to the original series give (using the above notation):

$$R_m = \sum_{n=m}^{\infty} a_n = -\frac{a_n}{A_0} - \left[\sum_{s=1}^{p-1}\prod_{k=1}^{s}\frac{1}{A_k}\left(1-\frac{\Delta}{A_{k-1}}\right)\right]a_m + \\ + \sum_{u=m}^{\infty}\prod_{s=0}^{p-1}\left(1-\frac{\Delta}{A_s}\right)a_n, \qquad (3.207)$$

where

$$A_0 = \lim_{n\to\infty}\frac{\Delta a_n}{a_n} \quad \text{and} \quad A_k = \lim_{n\to\infty}\frac{\Delta\prod_{s=0}^{k-1}\left(1-\frac{\Delta}{A_s}\right)a_n}{\prod_{s=0}^{k-1}\left(1-\frac{\Delta}{A_s}\right)a_n},$$

$$(k = 1, 2, 3, \ldots, p).$$

Euler's transformation. Given the power series

$$\sum_{n=1}^{\infty} \alpha_n x^n \qquad (3.208)$$

suppose that the following conditions are satisfied;

$$\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1\dagger \quad \text{and} \quad \lim_{n\to\infty}\frac{\Delta^k\alpha_{n+1}}{\Delta^k\alpha_n} = 1 \qquad (3.209)$$

$$(k = 1, 2, \ldots, p).$$

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† This guarantees the convergence of series (3.208) with $|x| < 1$ (see ref. 11, vol. II).

We can now apply to series (3.208) the transformation (3.207), which reduces to the following familiar Euler transformation:

$$\sum_{n=1}^{\infty} \alpha_n x^n = \frac{\alpha_1 x}{1-x} + \sum_{k=1}^{p-1} \Delta^k \alpha_1 \left(\frac{x}{1-x}\right)^{k+1} + \\ + \left(\frac{x}{1+x}\right)^p \sum_{n=1}^{\infty} (\Delta^p \alpha_n) x^n. \qquad (3.210)$$

EXAMPLE 51. Let $\alpha_n = P(n)$ be a polynomial of degree m; it is easily verified that conditions (3.209) are satisfied here for all $k \leqq m$, while we have in addition $\Delta^k \alpha_n = 0$ for any $k > m$. Now, on applying (3.210), we obtain, for $|x| < 1$:

$$\sum_{n=1}^{\infty} P(n) x^n = \frac{P(1)x}{1-x} + \sum_{k=1}^{m} \Delta^k P(1) \left(\frac{x}{1-x}\right)^{k+1}.$$

Improvement of the convergence of series corresponding to Gauss's test. Suppose that the series $\sum_{n=m}^{\infty} a_n$ is convergent where the conditions of Gauss's test are satisfied, i.e.

$$\frac{a_{n+1}}{a_n} = \frac{n^\lambda + p_1 n^{\lambda-1} + \varphi(n)}{n^\lambda + q_1 n^{\lambda-1} + \psi(n)},$$

where $\varphi(n)$ and $\psi(n)$ have orders lower than $n^{\lambda-1}$, and in addition, $q_1 - p_1 > 1$ (see § 1, sec. 5).

If we put $b_n = (n+1)a_{n+1} - na_n$, transformation (3.204) can be applied to the series $\sum_{n=m}^{\infty} a_n$. We have

$$\sum_{n=m}^{\infty} a_n = \frac{ma_m}{q_1 - p_1 - 1} + \\ + \frac{1}{q_1 - p_1 - 1} \sum_{n=m}^{\infty} \frac{p_1' n^{\lambda-1} + p_2' n^{\lambda-2} + \ldots + p_\lambda'}{n^\lambda + q_1 n^{\lambda-1} + \ldots + q_\lambda} a_n, \qquad (3.211)$$

where $p_1', p_2', \ldots, p_\lambda'$ are certain new coefficients not dependent on n. Transformation (3.204) for the series (3.211) can be repeated.

EXAMPLE 52. If transformation (3.211) is applied twice to the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

we get

$$S = \frac{2}{3} + 6\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}.$$

If we take into account only three terms of the last series, we get $S \approx 59/60$, while the exact sum is $S = 1$.

Example 53. Given the series

$$S = \sum_{n=m}^{\infty} a_n, \tag{3.212}$$

where

$$a_n = \prod_{k=0}^{n-1} \frac{(\alpha+k)(\beta+k)}{(k+1)(\gamma+k)} \quad \text{and} \quad \alpha+\beta-\gamma < 0,$$

we can apply transformation (3.211) several times. We obtain after p transformations of the series (3.212):

$$S = \frac{\alpha\beta}{\gamma(\gamma-\alpha-\beta)} + \\ + \frac{\alpha\beta}{\gamma(\gamma-\alpha-\beta)} \sum_{k=0}^{p-1} \prod_{s=0}^{k} \frac{(\gamma+s-\alpha)(\gamma+s-\beta)}{(\gamma+s+1)(\gamma+s+1-\alpha-\beta)} + \\ + \prod_{s=0}^{p} \frac{(\gamma+s-\alpha)(\gamma+s-\beta)}{\beta+s-\alpha-\beta} \sum_{n=m}^{\infty} \frac{a_n}{\prod_{s=0}^{p}(n+\gamma-s)}.$$

7. Abel's transformation

The identity

$$\sum_{k=1}^{n} \alpha_k\beta_k = \alpha_n\beta_n + \sum_{k=1}^{n-1} (\alpha_k - \alpha_{k+1})B_k, \tag{3.213}$$

where $B_k = \sum_{i=1}^{k} \beta_i$, is called *Abel's transformation*. It is the analogue of the formula for integration by parts of finite sums. On letting $k \to \infty$ in (3.213), we arrive at infinite sequences $\{\alpha_k\}$ and $\{\beta_k\}$. It follows from (3.213), provided $\{B_k\}$ is bounded and $\lim_{k\to\infty} \alpha_k = 0$, that

$$\sum_{k=1}^{\infty} \alpha_k\beta_k = \sum_{k=1}^{\infty} (\alpha_k - \alpha_{k+1})B_k, \tag{3.214}$$

This transformation can be utilized for improving the convergence of infinite series (see ref. 1).

EXAMPLE 54. Let us improve the convergence of the trigonometric series

$$\sum_{k=1}^{\infty} u_k \sin kx \quad \text{and} \quad \sum_{k=1}^{\infty} u_k \cos kx, \quad \text{where} \quad \lim u_k = 0,$$

with the knowledge that

$$\sum_{i=1}^{k} \sin ix = \frac{1}{2} \cot \frac{x}{2} - \frac{\cos\left(k+\frac{1}{2}\right)x}{2 \sin \frac{x}{2}}$$

and

$$\sum_{i=1}^{k} \cos ix = -\frac{1}{2} + \frac{\sin\left(k+\frac{1}{2}\right)x}{2 \sin \frac{x}{2}}.$$

On using transformation (3.214) after putting $\alpha_k = u_k$, $\Delta u_k = u_{k+1} - u_k$, we get:

$$\sum_{k=1}^{\infty} u_k \sin kx = \frac{u_1 \cot \frac{x}{2}}{2} + \frac{1}{2 \sin \frac{x}{2}} \sum_{k=1}^{\infty} \Delta u_k \cos\left(k+\frac{1}{2}\right)x; \tag{3.215}$$

$$\sum_{k=1}^{\infty} u_k \cos kx = -\frac{u_1}{2} - \frac{1}{2 \sin \frac{x}{2}} \sum_{k=1}^{\infty} \Delta u_k \sin\left(k+\frac{1}{2}\right)x. \tag{3.216}$$

Obviously, these transformations of trigonometric series can conveniently be employed in the case when the u_k tend to zero slowly and $\lim\limits_{k\to\infty} \Delta u_k/u_k = 0$. The improvement in the convergence will be the greater, the faster $\Delta u_k/u_k$ tends to zero.

If transformation (3.214) is applied a second time to series (3.215) and (3.216), we get

$$\sum_{k=1}^{\infty} u_k \sin kx = \left(u_1 - \frac{u_2}{2}\right) \cot \frac{x}{2} - \frac{1}{4 \sin^2 \frac{x}{2}} \sum_{k=1}^{\infty} \Delta^2 u_k \sin (k+1)x$$

and

$$\sum_{k=1}^{\infty} u_k \cos kx = -\frac{u_1}{2}+(u_1-u_2)\frac{\cos x}{4\sin^2\frac{x}{2}}\sum_{k=1}^{\infty} \Delta^2 u_k \cos (k+1)x,$$

where $\Delta^2 u_k = u_{k+2}-2u_{k+1}+u_k$ is a difference of the second order. For instance, if $u_k = 1/k$, we have $\Delta u_k = 1/k(k+1)$,

$\Delta^2 u_k = 2/k(k+1)(k+2)$, etc.

8. A. N. Krylov's method of improving the convergence of trigonometric series

If $f(x)$ satisfies Dirichlet's conditions, it can be expanded as a convergent Fourier series:

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{3.217}$$

where the Fourier coefficients a_n and b_n are of order $1/n$ if $f(x)$ is a function of bounded variation. Such a series is slowly convergent. It is possible to speed up the convergence by isolating the slowly convergent part. Let $f(x)$ have bounded derivatives of the kth order everywhere except for a finite number of points $x_1^{(0)}, x_2^{(0)}, \ldots, x_{m_0}^{(0)}$. At these points $f(x)$ has a discontinuity of the first kind, the size of the jump h_l being equal to

$$h_l = f(x_l^{(0)}+0)-f(x_l^{(0)}-0) \qquad (l = 1, 2, \ldots m_0). \tag{3.218}$$

Suppose that the jth derivative also has discontinuities of the first kind at the points $x_1^{(j)}, x_2^{(j)}, \ldots, x_{m_j}^{(j)}$, with jumps

$$h_l^{(j)} = f^{(j)}(x_l^{(j)}+0)-f^{(j)}(x_l^{(j)}-0) \tag{3.219}$$

$$(l = 1, 2, \ldots, m^j; \quad j = 1, 2, \ldots, k).$$

We can now write $f(x)$ in the form

$$f(x) = \sum_{l=1}^{m_0}\frac{1}{\pi}h_l^{(0)}\sigma_0(x-x_l^{(0)})+\sum_{l=1}^{m_1}\frac{1}{\pi}h_l^{(1)}\sigma_1(x-x_l^{(1)})+\ldots$$

$$\ldots+\sum_{l=1}^{m_k}\frac{1}{\pi}h_l^{(k)}\sigma_k(x-x_l^{(k)})+\varphi(x), \tag{3.220}$$

or

$$f(x) = \frac{1}{\pi}\sum_{j=0}^{k}\sum_{l=1}^{m_j} h_l^{(j)}\sigma_j(x-x_l^{(j)})+\varphi(x), \tag{3.221}$$

where

$$\sigma_0(u) = \sum_{n=1}^{\infty}\frac{\sin nu}{n} = \begin{cases} \dfrac{-\pi-u}{2} & \text{for} \quad -2\pi < u < 0, \\ \dfrac{\pi-u}{2} & \text{for} \quad 0 < u < 2\pi, \\ 0 & \text{for} \quad u = 0,\ 2\pi,\ -2\pi. \end{cases} \tag{3.222}$$

$$\sigma_1(u) = \int_0^{\infty}\sigma_0(u)\,du-\frac{\pi^2}{6} = -\sum_{n=1}^{\infty}\frac{\cos nu}{n^2} =$$

$$= \begin{cases} \dfrac{\pi^2}{12}-\dfrac{(\pi+u)^2}{4} & \text{for} \quad -2\pi \leqq u \leqq 0, \\ \dfrac{\pi^2}{12}-\dfrac{(\pi-u)^2}{4} & \text{for} \quad 0 \leqq u \leqq 2\pi. \end{cases} \qquad \text{(3:223}$$

Further integration gives us $\sigma_2(u)$:

$$\sigma_2(u) = -\sum_{n=1}^{\infty}\frac{\sin nu}{n^3} = \frac{3\pi u^2-2\pi^2u-u^3}{12} \quad (0 \leqq u \leqq 2\pi). \tag{3.224}$$

The function $\varphi(x)$ is continuous along with its first k derivatives, and its Fourier series

$$\varphi(x) = \frac{\alpha_0}{2}+\sum_{n=1}^{\infty}(\alpha_n\cos nx+\beta_n\sin nx)$$

is rapidly convergent, since the coefficients α_n and β_n have the order $1/n^{k+2}$. If we take into account the expansions (3.222)–(3.224), the series for $f(x)$ will be

$$f(x) = \sum_{l=1}^{m_0}\frac{1}{\pi}h_l^{(0)}\sum_{n=1}^{\infty}\frac{\sin n(x-x_l^{(0)})}{n}-\sum_{l=1}^{m_1}\frac{1}{\pi}h_l^{(1)}\sum_{n=1}^{\infty}\frac{\cos n(x-x_l^{(1)})}{n^2}-$$

$$-\sum_{l=1}^{m_2}\frac{1}{\pi}h_l^{(2)}\sum_{n=1}^{\infty}\frac{\sin n(x-x_l^{(2)})}{n^3}+\dots$$

$$\dots+\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}(\alpha_n\cos nx+\beta_n\sin nx). \tag{3.225}$$

The slowly convergent parts have thus been separated out from Fourier series (3.217).

We often obtain by some method the Fourier series for a function without knowing the function itself; here, if the series is slowly convergent, it is of little use for computing the values of the function, not to mention the values of its derivatives. It is often possible to improve the convergence in this case, by making use of familiar series for $\sigma(x-x_0)$, etc. This will be shown with the aid of an example (ref. 3).

EXAMPLE 55. We are given the Fourier series

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\cos n\frac{\pi}{2}}{n^2-1}\sin nx \qquad (0 \leqq x \leqq \pi). \tag{3.226}$$

We separate out the lowest power $1/n$ from the coefficient:

$$\frac{n}{n^2-1} = \frac{1}{n}+\frac{1}{n^3}+\frac{1}{n^5-n^3}.$$

The initial series can now be split into three:

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\frac{\pi}{2}}{n}\sin nx-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\frac{\pi}{2}}{n^3}\sin nx-$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\frac{\pi}{2}}{n^5-n^3}\sin nx. \tag{3.227}$$

The first two series are summable in the finite form:

$$S_1(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\frac{\pi}{2}}{n}\sin nx=$$

$$= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\left(x+\frac{\pi}{2}\right)+\sin n\left(x-\frac{\pi}{2}\right)}{n}=$$

$$= -\frac{1}{\pi}\left[\sigma_0\left(x+\frac{\pi}{2}\right)+\sigma_0\left(x-\frac{\pi}{2}\right)\right]; \tag{3.228}$$

$$S_2(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\frac{\pi}{2}}{n^3}\sin nx =$$

$$= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\left(x+\frac{\pi}{2}\right)+\sin n\left(x-\frac{\pi}{2}\right)}{n^3} =$$

$$= \frac{1}{\pi}\left[\sigma_2\left(x+\frac{\pi}{2}\right)+\sigma_2\left(x-\frac{\pi}{2}\right)\right]. \qquad (3.229)$$

On using (3.222) and (3.224), we get:

$$S_1(x) = \begin{cases} \frac{x}{\pi} & \left(0 \leqq x < \frac{\pi}{2}\right), \\ \frac{x-\pi}{.\pi} & \left(\frac{\pi}{2} < x \leqq \pi\right), \\ 0 & \left(x = \frac{\pi}{2}\right); \end{cases} \qquad (3.230)$$

$$S_2(x) = \begin{cases} -\frac{x^3}{6\pi}+\frac{\pi}{24}x & \left(0 \leqq x \leqq \frac{\pi}{2}\right), \\ -\frac{(x-\pi)^3}{6\pi}+\frac{\pi}{24}(x-\pi) & \left(\frac{\pi}{2} \leqq x \leqq \pi\right). \end{cases} \qquad (3.231)$$

Consequently,

$$f(x) = S_1(x)+S_2(x)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\frac{\pi}{2}}{n^3(n^2-1)}\sin nx. \qquad (3.232)$$

The remaining series is rapidly convergent (its coefficients are of order $1/n^5$) and the expression (3. 232) obtained enables us to evaluate easily the values of the function and of its first derivatives.

In the case when the series given above for σ_0, σ_1, σ_2 are insufficient for improving the convergence, the following series may prove useful:

$$\cos x\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+ \dots =$$

$$= -\mathrm{Re}[\ln|1-e^{ix}|] = -\ln\left|2\sin\frac{x}{2}\right| \qquad (3.233)$$

and its analogue:

$$\frac{\cos 2x}{1.2}+\frac{\cos 3x}{2.3}+\frac{\cos 4x}{3.4}+\dots$$

$$= (1-\cos x)\ln\left|2\sin\frac{x}{2}\right|+\left(\frac{\pi}{2}-\frac{x}{2}\right)\sin x+\cos x. \qquad (3.234)$$

By using these last series, we can improve the convergence of any series of the form

$$\sum_{n=1}^{\infty}\left[\left(A\left(\frac{1}{n}\right)\sin nx_0+B\left(\frac{1}{n}\right)\cos nx_0\right)\sin nx+\right.$$
$$\left.+\left(C\left(\frac{1}{n}\right)\sin nx_0+D\left(\frac{1}{n}\right)\cos nx_0\right)\cos nx\right],$$

where $A(1/n)$, $B(1/n)$, $C(1/n)$, $D(1/n)$ are analytic functions of $1/n$ for small values of the argument.

9. A. S. Maliev's method of improving the convergence of trigonometric series

The Fourier series of a function is slowly convergent if it has derivatives of all orders inside the interval $(0, 2\pi)$, while the function or one of its derivatives has different values at $x = 0$ and $x = 2\pi$. The following method was proposed by A. S. Maliev for obtaining a rapidly convergent trigonometric expansion of a function of this kind.

Let $f(x)$ be given in the interval $(0, \pi)$, where it has continuous derivatives up to and including the $(k-1)$th order and the kth derivative satisfies Dirichlet's conditions. This function can be expanded as a series in functions $\cos 2nx$ and $\sin 2nx$, or as a series in $\sin nx$ only or in $\cos nx$ only (depending on whether the continuation of $f(x)$ into the interval $(-\pi, 0)$ is even or odd). These series are in general slowly convergent. We can arrange for order $1/n^{k+1}$ in the Fourier coefficient if we first continue the function into the interval $(-\pi, 0)$ in such a way that, when continued periodically on to the whole of the real axis, the function has k derivatives, where the kth derivative satisfies Dirichlet's conditions.

For this, we take $f(x)$ in the interval $(-\pi, 0)$ in the form, say, of a $(2k-1)$th degree polynomial $\varphi(x)$, having, along with its derivatives up to the $(k-1)$th order, values at the points 0 and π, respectively, equal to the values of $f(x)$ and its derivatives at $x = 0$ and $x = \pi$. We can conveniently choose $\varphi(x)$ as

$$\varphi(x) = (x+\pi)^k\left[A_0+A_1x+\ldots+\frac{A_{k-1}}{(k-1)!}x^{k-1}\right]+$$
$$+x^k\left[B_0+B_1(x+\pi)+\ldots+\frac{B_{k-1}}{(k-1)!}(x+\pi)^{k-1}\right].$$

The coefficients $A_0, A_1, \ldots, A_{k-1}$; $B_0, B_1, \ldots, B_{k-1}$ are now determined successively from the equations:

$$\varphi(0) = \pi^k A_0 = f(0),$$
$$\varphi'(0) = k\pi^{k-1}A_0+\pi^k A_1 = f'(0),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\varphi^{(k-1)}(0) = C_{k-1}^{0}k(k-1)\ldots 2\pi A_0+C_{k-1}^{1}k(k-1)\ldots 3\pi^2 A_1+\ldots$$
$$\ldots+C_{k-1}^{k-2}k\pi^{k-1}A_{k-2}+C_{k-1}^{k-1}\pi^k A_{k-1} = f^{(k-1)}(0);$$
$$\varphi(-\pi) = (-\pi)^k B_0 = f(\pi),$$
$$\varphi'(-\pi) = k(-\pi)^{k-1}B_0+(-\pi)^k B_1 = f'(\pi),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\varphi^{(k-1)}(-\pi) = C_{k-1}^{0}k(k-1)\ldots 2(-\pi)B_0+$$
$$+C_{k-1}^{1}k(k-1)\ldots 3(-\pi)^2 B_1+\ldots$$
$$\ldots+C_{k-1}^{k-1}(-\pi)^k B_{k-1} = f^{(k-1)}(\pi).$$

After constructing the polynomial $\varphi(x)$, we get the function

$$\psi(x) = \begin{cases} \varphi(x) & \text{for} \quad -\pi \leqq x \leqq 0, \\ f(x) & \text{for} \quad 0 \leqq x \leqq \pi. \end{cases}$$

This function, continued periodically, has $k-1$ continuous derivatives and a kth derivative satisfying Dirichlet's conditions. The Fourier series for $\psi(x)$ is as convergent as $1/n^{k+1}$ and yields $f(x)$ in the interval $(0, \pi)$.

EXAMPLE 56 (ref. 3).

$$f(x) = x-\frac{\pi}{2} \qquad (0 \leqq x \leqq \pi).$$

Let $k = 3$ (for convergence of the order $1/n^4$). Now,

$$\varphi(x) = (x+\pi)^3\left(-\frac{1}{2\pi^2}-\frac{5}{2\pi^3}x-\frac{6}{\pi^4}x^2\right)+$$
$$+x^3\left[-\frac{1}{2\pi^2}-\frac{5}{2\pi^3}(x+\pi)-\frac{6}{\pi^4}(x+\pi)^2\right].$$

On expanding as a series the function

$$\psi(x) = \begin{cases} \varphi(x) & (-\pi \leqq x \leqq 0), \\ f(x) & (0 \leqq x \leqq \pi), \end{cases}$$

we get

$$f(x) = \frac{240}{\pi^3}\sum_{n=1,3,5\ldots}\left(\frac{1}{n^4}-\frac{12}{\pi^2 n^6}\right)\cos nx+\frac{1440}{\pi^4}\sum_{n=2,4,6,\ldots}\frac{1}{n^5}\sin nx.$$

CHAPTER IV

ORTHOGONAL SERIES AND ORTHOGONAL SYSTEMS

Introduction

A sequence of functions $\{f_n(x)\}$ is said to be *orthogonal* in the interval (a, b) if, for $i \neq j$,

$$\int_a^b f_i(x) f_j(x)\, dx = 0.$$

An important role is played in analysis by the representations of functions as orthogonal series:

$$\sum_{n=1}^{\infty} c_n f_n(x),$$

i.e. series in an orthogonal system of functions. Trigonometric series are the classical example of orthogonal series.

The theory of orthogonal series arose in connection with the solution of problems of mathematical physics by the so-called Fourier method. Linear integral equations with symmetric kernels may also be solved by means of orthogonal series.

The theory of orthogonal systems of functions has a remarkable analogy with the theory of orthogonal systems of vectors (see Chapter II, § 1). This analogy was observed long ago and has been reflected in the terminology. It led in the course of time to the conception of *Hilbert space* — the infinite-dimensional analogue of n-dimensional Euclidean spaces (see Chapter II), the role of orthogonal systems of vectors being played in Hilbert spaces by orthogonal systems of functions. To provide a valid basis for the theory of orthogonal systems of functions, the basic concepts of analysis had to be generalized; in particular, generalization of the concept of integral led to the *Lebesgue integral.*

The more elementary part of the theory, and especially the "computational" side of it is described in the present chapter. Unless there is a special proviso, the integral is understood in Reimann's sense throughout this chapter.

n-dimensional vectors can be interpreted as functions defined at n points; this interpretation is discussed in broad outline in § 1, sec. 1; it throws into greater relief the analogy with orthogonal series of functions. Biorthogonal systems of functions — the analogues of biorthogonal systems of vectors — are discussed in § 2, sec. 6. The first example of such a system was offered by P. L. Chebyshev in connection with interpolation problems.

Orthogonal systems of functions are connected with problems of approximation of complicated functions by simpler ones, when the measure of the closeness of two functions $f(x)$ and $\varphi(x)$ is provided by their square derivation:

$$\int_a^b [f(x)-\varphi(x)]^2\,dx.$$

The problem of the best approximation (from this point of view) of a function by polynomials led to the creation of the theory of orthogonal polynomials (the simplest system of orthogonal functions). The first example of an orthogonal system of polynomials was the system of Legendre polynomials. The general theory of orthogonal systems of polynomials is due to P. L. Chebyshev. This theory is described in § 3.

The classical systems of orthogonal polynomials are discussed from a unified view-point in § 4, sec. 1–4.

The later sections of § 4 are devoted to concrete systems of polynomials — those of Legendre, Jacobi, Chebyshev, Hermite, Laguerre (as also Chebyshev's analogue of Legendre polynomials for a finite number of points). Good uniform approximations of functions can be obtained in a number of cases with the aid of segments of series in orthogonal polynomials. Such approximations are obtained very often from segments of series in Chebyshev polynomials (see § 4, sec. 7).

§ 1. Orthogonal systems

1. Orthogonal systems of functions defined at n points

Logically, the simplest example of an orthogonal system of functions is provided by a system of functions that are defined at n points. We shall confine ourselves to functions of one variable (everything said can be generalized immediately to functions of several variables).

Let $x_1, x_2, \ldots, x_n$ be a finite set of numbers (or points of the real axis) and let $f(x_i)$ be functions defined at these points. Each such function can be regarded as a vector f with components $f_i = f(x_i)$ $(i = 1, 2, \ldots, n)$. The *length* or *norm* of such a vector is

$$\|f\| = \sqrt{\sum_{i=1}^{n} f_i^2}. \tag{4.1}$$

The set of these vectors (functions) forms an n-dimensional Euclidean space, which is denoted by the symbol $E_n(x_1, x_2, \ldots, x_n)$ (see Chapter II).

A *basis* in E_n $(x_1, x_2, \ldots, x_n)$ is any system of n linearly independent functions $f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ on the set $x_1, x_2, \ldots, x_n$.

2. Orthogonal systems in $E_n(x_1, x_2, \ldots, x_n)$

Functions $f(x_i)$ and $\varphi(x_i)$ are orthogonal on a set of points $x_1, x_2, \ldots, x_n$, if

$$(f, \varphi) \equiv \sum_{i=1}^{n} f_i\varphi_i = \sum_{i=1}^{n} f(x_i)\varphi(x_i) = 0, \tag{4.2}$$

which corresponds to the orthogonality of the vectors with coordinates f_i and $\varphi_i (i = 1, 2, \ldots, n)$.

A system of functions $f^{(k)}$ (x_i) $(k = 1, 2, \ldots, l)$ of E_n $(x_1, x_2, \ldots, x_n)$ is said to be *orthogonal* if

$$(f^{(j)}, f^{(k)}) = \begin{cases} \|f^{(j)}\|^2 > 0 & \text{for} \quad k = j, \\ 0 & \text{for} \quad k \neq j. \end{cases} \tag{4.3}$$

An orthogonal system of n functions is described as *complete*.

There exists in E_n $(x_1, x_2, \ldots, x_n)$ an infinite set of orthogonal systems of n functions $\{f^{(j)}\}$ $(j = 1, 2, \ldots, n)$; the functions of each such system form an *orthogonal basis* in space $E_n(x_1, x_2, \ldots, x_n)$. Every function $f(x_i)$ of $E_n(x_1, \ldots, x_n)$ is linearly expressible in terms of the functions of an orthogonal basis:

$$f(x_i) = \sum_{j=1}^{n} c_j f^{(j)}(x_i). \tag{4.4}$$

The coefficients c_j in (4.4) are called the *Fourier coefficients* of the function f in the system of functions $f^{(1)}, f^{(2)}, \ldots, f^{(n)}$, where

$$c_j = (f, f^{(j)}) = \sum_{i=1}^{n} f(x_i) f^{(j)}(x_i). \tag{4.5}$$

The coefficient c_j is the *projection* of the vector f on to the base vector $f^{(j)}$.

If $\|f^{(j)}\| = 1$ for all the $f^{(j)}$ $(j = 1, 2, \ldots, n)$ of an orthogonal basis, the basis is said to be *orthonormalized.* If the basis $f^{(j)}$ $(j = 1, 2, \ldots, n)$ is orthonormalized, we have for any function $f(x_i)$ of $E_n(x_1, x_2, \ldots, x_n)$:

$$\|f\|^2 = \sum_{j=1}^{n} c_j^2, \tag{4.6}$$

where the c_j are the Fourier coefficients of the function f in the orthonormalized system $\{f^{(j)}\}$ (this follows from (4.4) and the linear properties of the scalar product).

If an orthogonal normed $(\|f^{(j)}\| = 1)$ system is not complete, i.e. the number of its elements $l < n$, instead of equation (4.6) we have the inequality

$$\|f\|^2 \geqq \sum_{j=1}^{l} c_j^2. \tag{4.7}$$

Equation (4.6) and inequality (4.7) transform in the limit as $n \to \infty$ to *Parseval's equation* and *Bessel's inequality* respectively (see § 2, sec. 4).

3. The best mean square approximation

The *square approximation* of a function $f(x)$ by a function $\varphi(x)$ with respect to a system of points $x_1, x_2, \ldots, x_n$ is defined by

$$\|f-\varphi\| = \sqrt{\sum_{i=1}^{n} (f(x_i)-\varphi(x_i))^2}. \tag{4.8}$$

We shall discuss *generalized polynomials* of the lth order, i.e. functions of the form

$$\sum_{k=1}^{l} d_k f^{(k)}(x_i), \tag{4.9}$$

where $\{f^{(k)}\}$ $(k = 1, 2, \ldots, l;\ l < n)$ is an orthogonal system of functions of $E_n\,(x_1, x_2, \ldots, x_n)$.

THEOREM 1. *From among all the "generalized polynomials" of the l-th order* $(l \leqq n)$, *the best mean square approximation of a function* $f(x_i)$ *at the points* $x_1, x_2, \ldots, x_n$ *is given by a polynomial of the form*

$$\sum_{k=1}^{l} c_k f^{(k)}(x_i), \tag{4.10}$$

where c_k *are Fourier coefficients.*

This lth order polynomial coincides with the sum of the first l terms of sum (4.4). This best approximation vanishes when $l = n$.

4. Orthogonal systems of trigonometric functions

The most important examples of orthogonal systems on a finite set of points are first the *orthogonal systems of polynomials* introduced by Chebyshev (see § 4, sec. 11), and secondly, *orthogonal systems of trigonometric functions.*

EXAMPLE 1. The system of functions $\{\cos kx_i\}$ $(k = 0, 1, 2, \ldots, n)$ is orthogonal on the system of $2n$ points $x_i = i\pi/n$ $(i = 0, \pm 1, \pm 2, \ldots, \pm n-1, -n)$. Any even function $f(x_i)$, defined at these points, is expressible by the sum

$$f(x_i) = \frac{c_0}{2} + \sum_{k=1}^{n} c_k \cos kx_i, \tag{4.11}$$

where

$$c_k = \frac{1}{n} \sum_{i=-n}^{n-1} f(x_i) \cos kx_i.$$

EXAMPLE 2. The system of $n-1$ functions $\{\sin kx_i\}$ ($k = 1, 2, \ldots, n-1$) is orthogonal on the system of $2n-1$ points $x_i = i\pi/n$ ($i = \pm 1, \pm 2, \ldots, \pm n-1, -n$). Any odd function $\varphi(x_i)$, defined at these points, is expressible as

$$\varphi(x_i) = \sum_{k=1}^{n-1} c_k \sin kx_i, \tag{4.12}$$

where

$$c_k = \frac{1}{n} \sum_{i=-n}^{n-1} \varphi(x_i) \sin kx_i.$$

EXAMPLE 3. The system of $2n$ functions $\cos kx$, $\sin lx$, where $k = 0, 1, 2, \ldots, n$; $l = 1, 2, \ldots, n-1$, is orthogonal on the system of $2n$ points

$$x_i = \frac{i\pi}{n} \quad (i = 0, \pm 1, \pm 2, \ldots, \pm n-1, -n).$$

Any function $\psi(x_i)$, defined at these points, is expressible as

$$\psi(x_i) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx_i + b_k \sin kx_i). \tag{4.13}$$

where

$$a_k = \frac{1}{n} \sum_{i=-n}^{n-1} \psi(x_i) \cos kx_i,$$

$$b_k = \frac{1}{n} \sum_{i=-n}^{n-1} \psi(x_i) \sin kx_i.$$

Formula (4.13) reduces to (4.11) if $\psi(x)$ is an even function, and to (4.12) if $\psi(x)$ is odd.

Any periodic function $\psi(x_i)$, defined at all points $x_i = i\pi/n$, where i is any integer, is expressible by (4.13), or by (4.11) or (4.12) if it is even or odd.

These formulae, deduced by Euler and Lagrange, transform to (4. 42) and (4.43) as $n \to \infty$. They are at the basis of numerical methods of harmonic analysis

§ 2. General properties of orthogonal and biorthogonal systems

1. Orthogonality. Scalar (inner) product

The orthogonality of two vectors in n-dimensional Euclidean space is well known to consist in the fact that their scalar product vanishes (see Chapter II, § 1, sec. 3). The orthogonality of functions is defined by analogy with the orthogonality of vectors.

The scalar (inner) product of functions $f(x)$ *and* $g(x)$ *on* $[a, b]$ is defined by the expression

$$(f, g) = \int_a^b f(x)g(x)\,dx. \tag{4.14}$$

(We naturally assume here that $f(x).g(x)$ is integrable in $[a, b]$.) If

$$(f, g) = \int_a^b f(x)g(x)\,dx = 0,$$

$f(x)$ and $g(x)$ are said to be *orthogonal* on $[a, b]$.

EXAMPLE 4. The functions $\sin x$ and $\cos x$ are orthogonal on the segment $[0, \pi]$, since

$$\int_0^\pi \sin x \cos x\,dx = \left.\frac{\sin^2 x}{2}\right|_0^\pi = 0.$$

The *orthogonality of functions with respect to a weight function* is a further generalization of the concept of orthogonality.

Let $p(x)$ be a fixed non-negative function in $[a, b]$.

Any two functions $f(x)$ and $g(x)$, for which

$$\int_a^b f(x)g(x)p(x)\,dx = 0, \tag{4.15}$$

are said to be *orthogonal on the segment* $[a, b]$ *with respect to the weight* $p(x)$.

EXAMPLE 5. The functions $\sin(n \arccos x)$ and $\cos(n \arccos x)$ are orthogonal on $[-1, +1]$ with respect to the weight $1/\sqrt{1-x^2}$, since

$$\int_{-1}^{+1} \sin(n \arccos x) \cos(n \arccos x) \frac{1}{\sqrt{1-x^2}}\,dx =$$

$$= \int_0^\pi \sin nu \cdot \cos nu\,du = 0.$$

All the integrals considered in this chapter will be assumed to exist in Riemann's sense.

The concept of orthogonality can be generalized in a natural manner for functions $f(x)$, $g(x)$ and the weight $p(x)$, which are Lebesgue but not Riemann integrable.

Let $p(x)$ be non-negative in $[a, b]$, non-zero on a set of measure zero and Lebesgue integrable on $[a, b]$. Now,

$$\int_a^l p(x)\,dx > 0.$$

Let $f(x)$ be such that $\int_a^b f^3(x)\,dx$ exists in the Lebesgue sense. Then $\int_a^b f^2(x)p(x)\,dx$ also exists. We say in this case that $f(x)$ is *square integrable with respect to the weight* $p(x)$ on $[a, b]$ (in the Lebesgue sense), and we write $L^2_{p(x)}(a, b)$ for the set of such functions.

If $p(x) \equiv 1$, we use the simple notation $L^2(a, b)$.

For instance, the function $f(x) = x^{-1/3}$ belongs to $L^3(0, 1)$, whereas $f(x) = x^{-1/2}$ does not.

The following proposition holds: if $f(x)$ and $g(x)$ belong to $L^2_{p(x)}(a, b)$, then

$$\int_a^b f(x)g(x)p(x)\,dx$$

exists, and the *Cauchy–Bunyakovskii inequality* holds:

$$\int_a^b f(x)g(x)p(x)\,dx \leqq \sqrt{\int_a^b f^2(x)p(x)\,dx \int_a^b g^2(x)p(x)\,dx}$$

(this latter is a generalization of the corresponding inequality for vectors, see Chapter II, § 1, sec. 2).

The *scalar product* in $L^2_{p(x)}(a, b)$ of functions $f(x)$ and $g(x)$ is defined by the expression

$$(f, g) = \int_a^b f(x)g(x)p(x)\,dx.$$

The functions $f(x)$ and $g(x)$ of $L^2_{p(x)}(a, b)$ are said to be *orthogonal with respect to the weight* $p(x)$ if $(f, g) = 0$.

Let $\sigma(x)$ be a non-decreasing function in $[a, b]$. Let us consider the functions $f(x)$, for which the *Stieltjes integral* exists:

$$\int_a^b f^2(x)\,d\sigma(x).$$

$\left(\text{In this case } \int_a^b f(x)\,d\sigma(x) \text{ also exists.}\right)$ The set of such functions will be denoted by $L^2_{\sigma(x)}(a, b)$.

There exists for any two functions $f(x)$ and $g(x)$ of $L^2_{\sigma(x)}(a, b)$:

$$\int_a^b f(x)g(x)\,d\sigma(x) \tag{4.16}$$

and the corresponding Cauchy–Bunyakovskii inequality holds. Integral (4.16) is called the *scalar (inner) product of functions* $f(x)$ *and* $g(x)$ of $L^2_{\sigma(x)}(a, b)$.

The functions $f(x)$ and $g(x)$ are said to be *orthogonal with respect to the integral weight* $\sigma(x)$ *if*

$$(f, g) = \int_a^b f(x)g(x)\,d\sigma(x) = 0,$$

i.e. their scalar product vanishes.

In the case when $\sigma(x)$ has a finite number of growth points: $x_1, x_2, \ldots, x_n$, integral (4.16) becomes the finite sum

$$\sum_{i=1}^{n} f(x_i)g(x_i)\,[\sigma(x_i+0)-\sigma(x_i-0)]. \tag{4.17}$$

If all the jumps of $\sigma(x)$ are equal to 1, i.e.

$$\sigma(x_i+0)-\sigma(x_i-0) = 1$$

for all i, (4.17) becomes

$$\sum_{i=1}^{n} f(x_i)g(x_i). \tag{4.18}$$

The orthogonality of f and g is influenced here only by their values at a finite number of points — the growth points of $\sigma(x)$. In this connection, the functions f and g can be regarded as specified only at these points (as in § 1, sec. 1). In this case (4.18) is the ordinary scalar product of the vectors $\bar{f}$ and $\bar{g}$ of $E_n(x_1, \ldots, x_n)$. The case when $\sigma(x)$ has a finite number of growth points is singular inasmuch as it is only in this case that there exist only n linearly independent mutually orthogonal functions (see § 1, sec. 1).

The above concepts relating to the orthogonality of two functions are united by a common property: the inner product of the functions $f(x)$ and $g(x)$ is in each case equal to zero. The whole of our subsequent discussion (unless there is a special proviso) will be independent of the particular method by which the concept of inner product is introduced. We shall thus use in general the symbol (f, g) to denote the inner product of $f(x)$ and $g(x)$, i.e. this symbol may be defined in any of the ways described. The case when

$$(f, g) = \int_a^b f(x)g(x)\, d\sigma(x).$$

unites all the other definitions of scalar product. The case of a "differential" weight $p(x)$ is obtained from this when $\sigma(x)$ has an integrable derivative, and $d\sigma(x) = p(x)\, dx$, so that integral (4.16) becomes

$$\int_a^b f(x)g(x)p(x)\, dx.$$

Orthogonality without a weight is the case when $d\sigma(x) = dx$. The orthogonality of functions defined at a finite number of points corresponds to a finite number of growth points of $\sigma(x)$.

In future, we shall therefore use expression (4.16) for (f, g), where necessary.

The scalar product of the function $f(x)$ with $f(x)$, i.e. (f, f), is usually denoted by $||f||^2$ and $||f|| = \sqrt{(f, f)}$ is called the *norm* of the function. This quantity is the norm of $f(x)$ in the corresponding normed space.

The quantity

$$||f-g|| = \sqrt{\int_a^b (f-g)^2\, d\sigma(x)}$$

is called the *root square deviation* of functions $f(x)$ and $g(x)$. This is the measure in the accepted sense of the closeness of $f(x)$ and $g(x)$.

2. Orthogonal systems of Bessel functions, Haar functions, etc.

Let any two functions of the sequence

$$\varphi_1(x),\ \varphi_2(x),\ \ldots,\ \varphi_n(x),\ \ldots$$

be orthogonal on (a, b), i.e. $(\varphi_i, \varphi_j) = 0$ if $i \neq j$.

Such a sequence is described as an *orthogonal system of functions on* (a, b). If the system of functions

$$\varphi_1(x),\ \varphi_2(x),\ \ldots,\ \varphi_n(x),\ \ldots$$

is orthogonal and each of the functions differs from zero, the system of functions

$$\psi_1(x) = \frac{\varphi_1(x)}{\|\varphi_1(x)\|}, \quad \psi_2(x) = \frac{\varphi_2(x)}{\|\varphi_2(x)\|}, \quad \ldots, \quad \psi_n(x) = \frac{\varphi_n(x)}{\|\varphi_n(x)\|}, \quad \ldots$$

is also orthogonal. Every function of this system satisfies

$$\|\psi_i(x)\| = \sqrt{(\psi_i, \psi_i)} = 1. \tag{4.19}$$

Such systems are extremely convenient. They are described as *orthonormalized* (or *orthonormal*).

The orthogonal system

$$\varphi_1(x), \; \varphi_2(x), \; \ldots, \; \varphi_n(x), \; \ldots$$

is said to be complete in the space $L^2_{\sigma(x)}(a, b)$ if, among all the functions of $L^2_{\sigma(x)}(a, b)$, there is none that is non-zero and orthogonal to all the functions of the system

$$\varphi_1(x), \; \varphi_2(x), \; \ldots, \; \varphi_n(x), \; \ldots,$$

i.e. the fact that

$$\int_a^b f(x)\varphi_n(x)\, d\sigma(x) = 0$$

for all n, implies that

$$f(x) \equiv 0$$

(excluding possibly a set of measure zero).

In the case of a finite number of growth points of $\sigma(x)$, a complete system contains as many functions as there are growth points of $\sigma(x)$.

EXAMPLE 6. The system of functions

$$\sin x, \; \sin 2x, \; \ldots, \; \sin nx, \; \ldots \tag{4.20}$$

is orthogonal on $(0, \pi)$, since

$$\int_0^\pi \sin mx \sin nx\, dx = 0 \quad \text{for} \quad m \neq n.$$

The system is complete.

EXAMPLE 7. The system of functions

$$1, \; \cos x, \; \cos 2x, \; \ldots, \; \cos nx, \; \ldots \tag{4.21}$$

is orthogonal on $(0, \pi)$, since

$$\int_0^\pi \cos mx \cos nx \, dx = 0 \quad \text{for} \quad m \neq n.$$

The system is complete.

EXAMPLE 8. The system of functions

$$1, \ \sin x, \ \cos x, \ \sin 2x, \ \cos 2x, \ \ldots, \ \sin nx, \ \cos nx, \ \ldots \tag{4.22}$$

is orthogonal on $(0, 2\pi)$, since

$$\int_0^{2\pi} \sin mx \cos nx \, dx = 0 \quad \text{for} \quad \text{any} \quad m \text{ and } n,$$

$$\int_0^{2\pi} \sin mx \sin nx \, dx = 0 \quad \text{for} \quad m \neq n,$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = 0 \quad \text{for} \quad m \neq n$$

$$(m = 0, 1, 2, \ldots; \ n = 0, 1, 2, \ldots).$$

The system is complete.

The system of Bessel functions. Let $J_n(x)$ be the Bessel function of order n, and $\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots$ the positive roots of the equation

$$J_n(x) = 0$$

or of the equation

$$J_n'(x) = 0.$$

The system of functions

$$J_n(\lambda_1 x), \quad J_n(\lambda_2 x), \quad J_n(\lambda_3 x), \ \ldots, \ J_n(\lambda_i x), \ \ldots \tag{4.23}$$

is now orthogonal in $[0, 1]$ with respect to the weight x, since with $i \neq j$:

$$\int_0^1 J_n(\lambda_i x) J_n(\lambda_j x) x \, dx = 0,$$

The system (4.23) is complete.

The system of Haar functions. In recent times a use has been found for the *orthogonal system of Haar functions.* This system consists of piecewise constant functions $\chi_n^k(t)$, where

$$\left.\begin{aligned}
\chi_0^{(0)}(t) &= 1 \qquad \text{for} \quad t\in[0,\ 1];\\
\chi_0^{(1)}(t) &= \begin{cases} 1 & \text{for} \quad t\in\left[0,\ \frac{1}{2}\right),\\ -1 & \text{for} \quad t\in\left(\frac{1}{2},\ 1\right],\\ 0 & \text{for} \quad t=\frac{1}{2};\end{cases}\\
\chi_1^{(1)}(t) &= \begin{cases} \sqrt{2} & \text{for} \quad t\in\left[0,\ \frac{1}{4}\right),\\ -\sqrt{2} & \text{for} \quad t\in\left(\frac{1}{4},\ \frac{1}{2}\right],\\ 0 & \text{at other points};\end{cases}\\
\chi_1^{(2)}(t) &= \begin{cases} \sqrt{2} & \text{for} \quad t\in\left[\frac{1}{2},\ \frac{3}{4}\right),\\ -\sqrt{2} & \text{for} \quad t\in\left(\frac{3}{4},\ 1\right],\\ 0 & \text{at other points};\end{cases}\\
\chi_2^{(1)}(t) &= \begin{cases} 2 & \text{for} \quad t\in\left[0,\ \frac{1}{8}\right),\\ -2 & \text{for} \quad t\in\left(\frac{1}{8},\ \frac{1}{4}\right],\\ 0 & \text{at other points};\end{cases}\\
\chi_2^{(2)}(t) &= \begin{cases} 2 & \text{for} \quad t\in\left[\frac{1}{4},\ \frac{3}{8}\right),\\ -2 & \text{for} \quad t\in\left(\frac{3}{8},\ \frac{1}{2}\right],\\ 0 & \text{at other points};\end{cases}\\
\chi_2^{(3)}(t) &= \begin{cases} 2 & \text{for} \quad t\in\left[\frac{1}{2},\ \frac{5}{8}\right),\\ -2 & \text{for} \quad t\in\left(\frac{5}{8},\ \frac{3}{4}\right],\\ 0 & \text{at other points};\end{cases}
\end{aligned}\right\} \qquad (4.24)$$

$$\chi_2^{(4)}(t) = \begin{cases} 2 & \text{for } t \in \left[\frac{3}{4}, \frac{7}{8}\right), \\ -2 & \text{for } t \in \left(\frac{7}{8}, 1\right], \\ 0 & \text{at other points.} \end{cases} \tag{4.24}$$

In general,

$$\chi_n^{(1)}(0) = \sqrt{2^n}, \quad \chi_n^{(2^n)}(1) = -\sqrt{2^n}.$$

$$\chi_n^{(k)}(t) = \begin{cases} \sqrt{2^n} & \text{for } t \in \left[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right), \\ -\sqrt{2^n} & \text{for } t \in \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right], \\ 0 & \text{at other points } [0, 1]. \end{cases} \tag{4.25}$$

The Haar functions are generated from the "*polygonal functions*" of Schauder's basis. The orthogonality of the Haar system follows from the fact that, given the same n,

$$\chi_n^{(j)}(t)\chi_n^{(k)}(t) \equiv 0 \quad \text{for} \quad j > k \geqq 1,$$

so that

$$(\chi_n^{(j)}, \chi_n^{(k)}) = 0.$$

Given different subscripts m and n, where $m > n$, and any j and k the sub-intervals, where $\chi_m^{(k)}(t) \neq 0$ are wholly contained in the intervals of constancy of $\chi_n^j(t)$ for $m > n$, and

$$\int_0^1 \chi_m^{(k)}(t)\chi_n^{(j)}(t)\,dt = \lambda \int_{\frac{2k-2}{2^{m+1}}}^{\frac{2k}{2^{m+1}}} \chi_m^{(k)}(t)\,dt = \lambda\left(\frac{\sqrt{2^m}}{2^{m+1}} - \frac{\sqrt{2^m}}{2^{m+1}}\right) = 0,$$

where λ is the value of $\chi_n^{(j)}$ on the interval of constancy.

The system of Haar functions has an important approximation property: the Fourier series (see Chapter III) of any continuous function with respect to the Haar system is uniformly convergent to it in [0, 1].

The concept of orthogonality of functions of one variable on a segment may be generalized in a natural manner to functions of several variables, defined in some domain.

The functions $f(P)$ and $g(P)$ of a point P of n-dimensional space are said to be *orthogonal with respect to a domain D* of this space if

$$\int_D f(P)g(P)\,dP = 0. \tag{4.26}$$

A similar definition can be given of the *orthogonality of $f(P)$ and $g(P)$ with respect to the weight $k(P)$*:

$$\int_D f(P)g(P)k(P)\,dP = 0. \tag{4.27}$$

EXAMPLE 9. The problem of the vibrations of a plane clamped membrane leads to the differential equation

$$\Delta U+\lambda U = 0$$

(where Δ is *Laplace's operator*) with the boundary condition $U = 0$. The eigenfunctions of this equation, corresponding to different values of λ, are orthogonal:

$$\int_G\int U_i(x, y)U_k(x, y)\,dx\,dy = 0$$

EXAMPLE 10. Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ be a system of functions orthogonal on the segment $[a, b]$. The system of functions

$$U_{i,j}(x, y) = \varphi_i(x)\varphi_j(y) \tag{4.28}$$

is now orthogonal with respect to area on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$.

For $\int_a^b\int_a^b \varphi_i(x)\varphi_j(y)\varphi_k(x)\varphi_l(y)\,dx\,dy =$

$$= \int_a^b \varphi_i(x)\varphi_k(x)\,dx \int_a^b \varphi_j(y)\varphi_l(y)\,dy = 0,$$

if either $i \neq k$ or $j \neq l$, i.e. if $U_{i,j}(x, y)$ and $U_{k,l}(x, y)$ are different functions of system (4.28).

Systems of the form (4.28) are utilized, for example, in the expansion of the symmetric kernel of the integral equation

$$\varphi(x) = \lambda\int_a^b K(x, s)\varphi(s)\,ds+f(x) \tag{4.29}$$

as a series in the eigenfunctions of the equation.

If $\{\varphi_i(x)\}$ is a normed system of eigenfunctions of (4.29), corresponding to the eigenvalues $1/\lambda_i$, while $\{O_i(x)\}$ is a system of eigenfunctions corresponding to the zero eigenvalue, the system of functions (where i and j are arbitrary):

$$\varphi_i(x)\varphi_j(s), \quad O_i(x)\varphi_j(s), \quad \varphi_i(x)O_j(s), \quad O_i(x)O_j(s), \tag{4.30}$$

is orthogonal on the square ($a \leqq x \leqq b;\ a \leqq s \leqq b$), since the system $\{\varphi_i(x), O_j(x)\}$ is orthogonal on $[a, b]$.

System (4.30) is complete in the space of functions square integrable on ($a \leqq x \leqq b;\ a \leqq s \leqq b$). Hence the square integrable function $K(x, s)$ can be expanded as a series in this system. The Fourier coefficients of the function $K(x, s)$ with respect to the functions $\varphi_i(x)\varphi_j(s)$ for $i \neq j$, $O_i(x)\varphi_j(s)$ and $\varphi_i(x)O_j(s)$ vanish for any i and j, while with respect to the functions $\varphi_i(x)\varphi_i(s)$ they are equal to $1/\lambda_i$, by virtue of the relationships

$$\int_a^b\int_a^b K(x, s)\varphi_i(x)\varphi_j(s)\,dx\,ds = \int_a^b \varphi_i(x)\,dx\int_a^b K(x, s)\varphi_j(s)\,ds =$$
$$= \frac{1}{\lambda_j}\int_a^b \varphi_i(x)\varphi_j(x)\,dx = 0, \quad \text{if} \quad i \neq j;$$

$$\int_a^b\int_a^b K(x, s)\varphi_i(x)O_j(s)\,dx\,ds = \int_a^b \varphi_i(x)\,dx\int_a^b K(x, s)O_j(s)\,ds = 0,$$

$$\int_a^b\int_a^b K(x, s)\varphi_i(x)\varphi_i(s)\,dx\,ds = \int_a^b \varphi_i(x)\,dx\int_a^b K(x, s)\varphi_i(s)\,ds =$$
$$= \frac{1}{\lambda_i}\int_a^b \varphi_i(x)\varphi_i(x)\,dx = \frac{1}{\lambda_i}.$$

Hence $$K(x, s) = \sum_{i=1}^{\infty}\frac{1}{\lambda_i}\varphi_i(x)\varphi_i(s).$$

The concepts of orthogonality of functions with respect to surfaces and curves can be treated in the same way as the above orthogonality of functions in a domain D of n-dimensional space. For instance, the *spherical Laplace functions* are orthogonal with respect to the surface of a sphere.

3. Linear independence. The process of orthogonalization

A system of n functions is said to be *linearly independent* if, given any system of n numerical factors, for which

$$\sum_{i=1}^{n} \lambda_i^2 > 0,$$

the function $\lambda_1\varphi_1(x)+\lambda_2\varphi_2(x)+\ldots+\lambda_n\varphi_n(x)$ is non-zero. If there exists a system of factors λ_i such that

$$\lambda_1\varphi_1(x)+\lambda_2\varphi_2(x)+\ \ldots\ +\lambda_n\varphi_n(x) \equiv 0, \tag{4.31}$$

where at least one λ_i differs from zero, the system of functions $\varphi_1(x)$, $\varphi_2(x), \ldots, \varphi_n(x)$ is said to be *linearly dependent*. (In the case of functions of $L_p^2(a, b)$, equation (4.31) can hold for a linearly independent system on a set of measure zero.)

The concept of linear independence of functions is similar to the concept of linear independence of vectors of n-dimensional vector space (see Chapter II, § 1, sec. 4).

Let the inner product (φ_i, φ_j) be defined for any pair $\varphi_i(x)$, $\varphi_j(x)$ of the function system $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$.

In analogy with the Gram determinant for a vector system (see Chapter II, § 1, sec. 5), we introduce the determinant

$$\varDelta_n = \begin{vmatrix} (\varphi_1, \varphi_1) & (\varphi_1, \varphi_2) & \ldots & (\varphi_1, \varphi_n) \\ (\varphi_2, \varphi_1) & (\varphi_2, \varphi_2) & \ldots & (\varphi_2, \varphi_n) \\ \cdots & \cdots & \cdots & \cdots \\ (\varphi_n, \varphi_1) & (\varphi_n, \varphi_2) & \ldots & (\varphi_n, \varphi_n) \end{vmatrix}, \tag{4.32}$$

which is called the *Gram determinant of the function system.*

Properties of the Gram determinant. 1°. The Gram determinant $\varDelta_n \geqq 0$ for any system of functions.

2°. Let the system of function, $\varphi_1, \varphi_2, \ldots, \varphi_n$ be defined on the segment $[a, b]$ and let the interval (a_1, b_1) be contained in (a, b).

Now, if $\varDelta_n$ is the Gram determinant of the system in $[a, b]$ and (φ_i, φ_j) is defined by

$$(\varphi_i, \varphi_j) = \int_a^b \varphi_i(x)\varphi_j(x)\, d\sigma(x)$$

while Δ'_n is the Gram determinant of the same system in $[a_1, b_1]$, we have

$$\Delta_n \geqq \Delta'_n.$$

This property also holds for the Gram determinant of functions of several variables in a domain Q: the Gram determinant can only diminish when the domain diminishes.

THEOREM 2. *The necessary and sufficient condition for a finite system of functions to be linearly independent is that its Gram determinant be different from zero.*

THEOREM 3. *Every finite orthogonal system of functions is linearly independent.*

Theorem 3 is an obvious consequence of Theorem 2, since the Gram determinant of an orthogonal system of functions is equal to the product of the squares of the norms of all the functions of the system, each of which is greater than zero.

An infinite sequence of functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ is said to be *linearly independent* if any finite sub-system of it is linearly independent.

Given the system $\varphi_1, \varphi_2, \ldots, \varphi_n$ of linearly independent functions and some function $f(x)$, let us consider the problem of the best root square approximation of $f(x)$ by means of linear combinations of the system, i.e. from all the linear combinations $\sum_{i=1}^{n} d_i\varphi_i$ we find the one that gives a minimum for the expression

$$\left\| f - \sum_{i=1}^{n} d_i\varphi_i \right\|^2.$$

The coefficients of the best linear combination are found from the system of linear equations

$$\frac{\partial}{\partial d_k}\left(\left\| f - \sum_{i=1}^{n} d_i\varphi_i \right\|^2 \right) = 0 \qquad (k = 1, 2, \ldots, n),$$

i.e.

$$\left.\begin{array}{l} d_1(\varphi_1, \varphi_1) + d_2(\varphi_2, \varphi_1) + \ldots + d_n(\varphi_n, \varphi_1) = (f, \varphi_1), \\ d_1(\varphi_1, \varphi_2) + d_2(\varphi_2, \varphi_2) + \ldots + d_n(\varphi_n, \varphi_2) = (f, \varphi_2), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ d_1(\varphi_1, \varphi_n) + d_2(\varphi_2, \varphi_n) + \ldots + d_n(\varphi_n, \varphi_n) = (f, \varphi_n). \end{array}\right\} \quad (4.33)$$

The determinant of this system is the Gram determinant; in view of the linear independence of the initial system, it differs from zero and (4.33) has a unique solution. Obviously, it is easiest to find the coefficients $d_1, d_2, \ldots, d_n$ from (4.33) if the initial system is orthogonal. In this case (4.33) becomes the system

$$d_k(\varphi_k, \varphi_k) = (f, \varphi_k) \qquad (k = 1, 2, \ldots, n). \tag{4.34}$$

By starting out from any (finite or infinite) *linearly independent* system of functions (or vectors), an orthogonal system can be formed, the elements of which are certain linear combinations of the elements of the initial system. Let us consider the process of forming such a system. It is called a *Schmidt orthogonalization process.*

Suppose we are given the linearly independent system

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots .$$

We put

$$\omega_1(x) = \frac{\varphi_1(x)}{(\varphi_1, \varphi_1)^{1/2}}.$$

Let

$$\psi_2 = \varphi_2(x) - c_{21}\omega_1(x).$$

The coefficient c_{21} can be chosen so that $(\psi_2, \omega_1) = 0$, by putting

$$c_{21} = \frac{(\varphi_2, \omega_1)}{(\omega_1, \omega_1)} = (\varphi_2, \omega_1).$$

Having chosen such a c_{21}, we put

$$\omega_2(x) = \frac{\psi_2(x)}{(\psi_2, \psi_2)^{1/2}}.$$

Here $(\psi_2, \psi_2) \neq 0$, since otherwise $\psi_2(x) \equiv 0$, and this would mean that φ_1 and φ_2 are linearly dependent.

Suppose that functions $\omega_1, \omega_2, \ldots, \omega_{n-1}$ have already been found. We now put

$$\psi_n(x) = \varphi_n(x) - \sum_{i=1}^{n-1} c_{ni}\omega_i(x),$$

here the numbers c_{ni} are chosen so that $\psi_n(x)$ is orthogonal to all the functions $\omega_1(x), \ldots, \omega_{n-1}(x)$. We must take for this: $c_{ni} = (\varphi_n, \omega_i)$. We next put $\omega_n(x) = \psi_n(x)/(\psi_n, \psi_n)^{1/2}$, where $(\psi_n, \psi_n) \neq 0$ in view of the linear independence of the functions $\varphi_1, \ldots, \varphi_{n-1}$. This process can be continued indefinitely.

The system $\omega_1(x), \ldots, \omega_n(x), \ldots$ thus formed will be *orthonormalized*

The process of orthogonalization of a vector system is precisely similar.

In the case of a vector system in n-dimensional space, or of a system of functions defined at n points, such a system contains not more than n elements.

4. Fourier coefficients. Closed systems

Let

$$\omega_1(x),\ \omega_2(x),\ \ldots,\ \omega_n(x),\ \ldots \tag{4.35}$$

be an orthonormal system of functions. Given a function $f(x)$, suppose that its inner product exists with any function of the system. The numbers $c_i = (f, \omega_i)$ are called the *Fourier coefficients* of $f(x)$ with respect to system (4.35) In the particular case when (4.35) is a trigonometric system, the c_i are the familiar Fourier coefficients of the theory of Fourier series.

The Fourier coefficients of functions are the analogues of the projections of a vector on to the vectors of an orthonormal vector system. If all the Fourier coefficients of a function exist, we can formally construct the series

$$\sum_{k=1}^{\infty} c_k \omega_k(x). \tag{4.36}$$

This series is called (in analogy with the corresponding series with respect to the trigonometric system) the *Fourier series of $f(x)$ with respect to the system $\omega_1(x), \ldots, \omega_n(x), \ldots$* .

Bessel's inequality and Parseval's equation. We can write *Bessel's inequality* for any $f(x)$, for which (f, f) and all the Fourier coefficients exist:

$$\sum_{i=1}^{n} c_i^2 \leq (f, f), \tag{4.37}$$

from which it follows that the series $\sum_{i=1}^{\infty} c_i^2$ is convergent and

$$\sum_{i=1}^{\infty} c_i^2 \leqq (f, f). \tag{4.38}$$

In the particular case when $\sum_{n=1}^{\infty} c_i^2 = (f, f)$, (4.38) is known as *Parseval's equation.*

Let

$$\omega_1(x),\ \omega_2(x),\ \ldots,\ \omega_n(x),\ \ldots \tag{4.39}$$

be an orthonormal system of functions on (a, b), and $f(x)$ a function defined on (a, b) for which (f, f) exists.

When considering the problem of the best *root square approximation* of $f(x)$ by the linear combinations

$$\sum_{i=1}^{n} d_i\omega_i(x),$$

we arrived (see sec. 3 of this article) at an expression for the coefficients of orthogonal system $\{\omega_i(x)\}$ giving the best approximation (4.34):

$$d_k = \frac{(f, \omega_k)}{(\omega_k, \omega_k)},$$

which yields in the case of a normed system:

$$d_k = (f, \omega_k) = c_k.$$

From this there follows:

THEOREM 4. *Of all the linear combinations* $\sum_{i=1}^{n} d_i\omega_i(x)$, *the finite Fourier sum*

$$\sum_{i=1}^{n} c_i\omega_i(x)$$

has the least root square deviation from $f(x)$ *(i.e. the linear combination for which* $d_i = c_i$*).*

The orthonormal system $\omega_1(x), \ldots, \omega_n(x), \ldots$ is said to be *closed* if Parseval's equation is satisfied

$$\sum_{i=1}^{\infty} c_i^2 = \int_a^b f^2(x)\, d\sigma(x) \tag{4.40}$$

for any function $f(x)$ of $L^2_{\sigma(x)}(a, b)$. In this case,

$$\int_a^b f^2(x)\, d\sigma(x) - \sum_{i=1}^{n} c_i^2 \to 0 \quad \text{as} \quad n \to \infty,$$

which implies, in view of the equation

$$\int_a^b f^2(x)\, d\sigma(x) - \sum_{i=1}^{n} c_i^2 = \int_a^b \left[f(x) - \sum_{i=1}^{n} c_i \omega_i(x) \right]^2 d\sigma(x)$$

that the root square deviation

$$\int_a^b \left[f(x) - \sum_{i=1}^{n} c_i \omega_i(x) \right]^2 d\sigma(x) \tag{4.41}$$

tends to zero as $n \to \infty$.

If (4.41) tends to zero as $n \to \infty$, we say that the Fourier series

$$\sum_{i=1}^{\infty} c_i \omega_i(x)$$

is *convergent in the mean* to $f(x)$.

Consequently, if $f(x)$ belongs to $L^2_{\sigma(x)}(a, b)$, its Fourier series with respect to any *closed* system is convergent to it in the mean.

EXAMPLE 11. The trigonometric system (4.22) is closed in (0, 2π.)

EXAMPLE 12.

THEOREM 5 (V. A. Steklov). *The set of eigenfunctions of the Sturm–Liouville equation forms a closed system.*

It was in the context of this theorem that V. A. Steklov introduced the concept of closure of a system.

It follows from Parseval's equation that every orthogonal system, closed in $L^2_{\sigma(x)}(a,b)$, is complete, and every complete orthogonal system in $L^2_{\sigma(x)}(a, b)$ is closed.

5. Fourier series in a trigonometric system†

The problem of representing an arbitrary function $f(x)$ by a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{4.42}$$

---

† See Chapter III for more detailed information on trigonometric Fourier series.

arose in the middle of the eighteenth century in connection with the problem of the vibrations of a string. This problem occupied all the greatest mathematicians of this period. The essential step in the solution of the problem was taken by Fourier, who established† that, by virtue of the orthogonality of the trigonometric system, the coefficient of series (4.42) can be expressed in the form:

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \tag{4.43}$$

(hence the name *Fourier coefficients* in the case of any orthogona- system).

Fourier stated the theorem to the effect that a (graphically) arbitrary function can be represented by a Fourier series.

Historically, the first theorem on the convergence of Fourier series was:

THEOREM 6 (Dirichlet, 1829). *Any function that admits of integration in any interval in* (0, 2π) *and does not have an infinity of extrema in the interval can be expanded as a Fourier series, which is convergent at every point x to the value*

$$\frac{f(x+0)+f(x-0)}{2}.$$

The crucial point in Dirichlet's proof was the representation of the partial sum $s_n(x)$ of the Fourier series and of the remainder in the form of integrals:

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\frac{\sin\dfrac{2n+1}{2}(x-\alpha)}{\sin\left(\dfrac{x-\alpha}{2}\right)}\,d\alpha$$

---

† This had been noticed before Fourier by Euler.

and

$$f(x)-s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)-f(\alpha))\,\frac{\sin\frac{2n+1}{2}(x-\alpha)}{\sin\left(\frac{x-\alpha}{2}\right)}\,d\alpha,$$

together with the following observations:

(1) if $0 < c < \frac{1}{2}\pi$, then as $n \to \infty$,

$$\int_0^c \varphi(\beta)\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta \to \frac{\pi}{2}\varphi(0);$$

(2) if $0 < b < c \leqq \frac{1}{2}\pi$ and $\varphi(\beta)$ is monotonic, then

$$\int_b^c \varphi(\beta)\frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta \to 0 \quad \text{as} \quad n \to \infty.$$

A rather more general theorem may be proved by the same method.

THEOREM 7. *If $f(x)$ is a function of bounded variation, its Fourier series is convergent at every point x to the value $\frac{1}{2}\,[f(x+0)+f(x-0)]$. If, in addition, $f(x)$ is continuous at every point of some interval, the Fourier series is uniformly convergent in the interval.*

THEOREM 8 (Lebesgue's Test). *Let*

$$\varphi(t) = \varphi_x(t) = f(x+t)+f(x-t)-2f(x)$$

and

$$\Phi_x(h) = \int_0^h |\varphi_x(t)|\,dt.$$

If

$$\frac{\Phi_x(h)}{h} \to 0 \quad \textit{as} \quad h \to 0$$

and

$$\int_\eta^\pi \frac{|\varphi(t)-\varphi(t+\eta)|}{t}\,dt \to 0 \quad \textit{as} \quad \eta \to 0,$$

the Fourier series of $f(x)$ is convergent to $f(x)$ at the point x.

THEOREM 9 (Dini–Lipschitz). *If $f(x)$ is continuous and its modulus of continuity $\omega(\delta)$ satisfies the condition*

$$\omega(\delta)\ln\frac{1}{\delta} \to 0 \quad \text{as} \quad \delta \to 0,$$

the Fourier series of $f(x)$ is uniformly convergent.

THEOREM 10 (de la Vallée-Poussin). *If the function*

$$\chi(t) = \frac{1}{t}\int_0^t \varphi_x(t)\,dt$$

is of bounded variation in some interval with left-hand end point $t = 0$ and $\chi(t) \to 0$ as $t \to 0$, the Fourier series of $f(x)$ is convergent at the point x to the value $f(x)$.

THEOREM 11 (Hardy). *If*

$$f(x+h)-f(x) = O\left[\left(\ln\frac{1}{|h|}\right)^{-1}\right]$$

and the coefficients c_n of the series have magnitudes of the order $O(n^{-\delta})$, where $\delta > 0$, the Fourier series is convergent at the point x to the value $f(x)$ (see Chapter I, § 3, sec. 8 regarding the symbols $o(n)$ and $O(n)$).

6. Biorthogonal systems of functions

A *biorthogonal system of functions* on (a, b) is a system consisting of two sequences

$$\left.\begin{array}{l}\varphi_0(x),\ \varphi_1(x),\ \varphi_2(x),\ \ldots,\ \varphi_n(x),\ \ldots \\ \psi_0(x),\ \psi_1(x),\ \psi_2(x),\ \ldots,\ \psi_n(x),\ \ldots\end{array}\right\} \tag{4.44}$$

satisfying the following condition on (a, b):

$$(\varphi_i, \psi_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta (see Chapter II, § 1, sec. 2). Here, the inner product (φ_i, ψ_j) is either $\int_a^b \varphi_i\psi_j\,dx$ or $\int_a^b \varphi_i\psi_j\,d\sigma(x)$. In the latter case we speak of a *system of functions, biorthogonal with respect to the weight* $\sigma(x)$ or $p(x)$, if $d\sigma(x) = p(x)\,dx$.

Given any function, for which the inner products encountered later exist, we can associate with it the series

$$f(x) \sim \sum_{k=0}^{\infty} A_k \varphi_k(x), \tag{4.45}$$

where

$$A_k = (f, \psi_k),$$

and consider the question of the convergence of the series and the possibility of using segments of it for approximation to the function.

EXAMPLE 13. *Chebyshev's biorthogonal system* on $(-1, 1)$ with respect to the weight $p(x) = 1$ is

$$\left.\begin{aligned} \varphi_k(x) &= \operatorname{sign} \frac{\sin(k+1)\varphi}{\sin\varphi}, \\ \psi_k(x) &= \frac{1}{2} \sum_{d/k+1} \frac{\mu(d)}{d} \, \frac{\sin \dfrac{k+1}{d}\varphi}{\sin\varphi}, \\ \text{where} \qquad x &= \cos\varphi \qquad (k = 0, 1, 2, \ldots), \end{aligned}\right\} \tag{4.46}$$

while d in (4.46) runs over all odd divisors of $k+1$, including $d = 1$ and $d = k+1$ if this is odd; the function sign x was introduced in Chapter I, § 2, sec. 1; $\mu(d)$ is the *Möbius function*, i.e.

$\mu(1) = 1$;

$\mu(a) = 0$, if a is divisible by a square different from unity;

$\mu(a) = (-1)^r$, if a is not divisible by a square different from unity, and r is the number of prime divisors of a differing from 1.

The functions $\varphi_k(k)$ of this system are piecewise constant, while $\psi_k(x)$ are polynomials in x having k simple zeros in $(-1, +1)$. The first few functions of this system are given below:

$$\varphi_0 = 1, \qquad \psi_0 = \frac{1}{2},$$

$$\varphi_1(x) = \begin{cases} -1 & \text{for} \quad -1 < x < 0, \\ +1 & \text{for} \quad 0 < x < 1, \end{cases} \qquad \psi_1(x) = x,$$

$$\varphi_2(x) = \begin{cases} -1 & \text{for} \quad -1 < x < -\dfrac{1}{2}, \\ +1 & \text{for} \quad -\dfrac{1}{2} < x < \dfrac{1}{2}, \\ -1 & \text{for} \quad \dfrac{1}{2} < x < 1. \end{cases} \qquad \psi_2(x) = 2\left(x^2 - \frac{1}{3}\right),$$

$$\varphi_3(x) = \begin{cases} -1 & \text{in} \quad \left(-1, \ -\dfrac{1}{\sqrt{2}}\right), \\ +1 & \text{in} \quad \left(-\dfrac{1}{\sqrt{2}}, \ 0\right), \\ -1 & \text{in} \quad \left(0, \ \dfrac{1}{\sqrt{2}}\right), \\ +1 & \text{in} \quad \left(\dfrac{1}{\sqrt{2}}, \ 1\right), \end{cases} \qquad \psi_3(x) = 2x(2x^2-1).$$

EXAMPLE 14. *Markov's biorthogonal system* in $(-1, +1)$ with respect to the weight $p(x) = 1/\sqrt{1-x^2}$ is

$$\left.\begin{aligned} \varphi_0 &= 1, & \psi_0 &= \frac{1}{\pi}, \\ \varphi_k &= \operatorname{sign} \cos k\varphi, & \psi_k &= \frac{1}{2}\sum_d \frac{(-1)^h}{d} \cos\frac{k\varphi}{d}, \end{aligned}\right\} \tag{4.47}$$

where $x = \cos\varphi$, d in (4.47) runs over all add divisors of the number k that do not contain square factors, and h is the number of prime factors of the form $4m+1$ contained in d $(m \neq 0)$ (see Chapter I, § 2, sec. 1, regarding the function sign x).

Here, as in the Chebyshev system, $\varphi_k(x)$ are piecewise constant, while $\psi_k(x)$ are polynomials in x. The first few functions of this system are given below:

$$\varphi_0 = 1, \qquad \psi_0 = \frac{1}{\pi},$$

$$\varphi_1(x) = \begin{cases} -1 & \text{in} \quad (-1, 0,) \\ +1 & \text{in} \quad (0, 1) \end{cases} \qquad \psi_1(x) = \frac{x}{2},$$

$$\varphi_2(x) = \begin{cases} +1 & \text{in} \quad \left(-1, \ -\dfrac{1}{\sqrt{2}}\right), \\ -1 & \text{in} \quad \left(-\dfrac{1}{\sqrt{2}}, \ \dfrac{1}{\sqrt{2}}\right), \\ +1 & \text{in} \quad \left(\dfrac{1}{\sqrt{2}}, \ 1\right), \end{cases} \qquad \psi_2(x) = \frac{1}{2}(2x^2-1),$$

$$\varphi_3(x)=\begin{cases}-1 & \text{in } \left(-1,\ -\frac{\sqrt{3}}{2}\right),\\ +1 & \text{in } \left(-\frac{\sqrt{3}}{2},\ 0\right),\\ -1 & \text{in } \left(0,\ \frac{\sqrt{3}}{2}\right),\\ +1 & \text{in } \left(\frac{\sqrt{3}}{2},\ 1\right),\end{cases} \qquad \psi_3(x)=2x\left(x^2-\frac{2}{3}\right).$$

Series in Chebyshev and Markov functions are similar to series in sines only or in cosines only.

On passing from polynomials and the interval $(-1, +1)$ to trigonometric sums and the interval $(-\pi, +\pi)$, we can combine both systems into one, which generates series corresponding to trigonometric series of a general type.

A biorthogonal system is called a *Chebyshev–Markov system with respect to the weight* $\sigma(x)$, if $\varphi_k(x)$ is a polynomial of degree k, and

$$\psi_k(x) = \operatorname{sign} Q_k(x),$$

where $Q_k(x)$ is a polynomial of degree k.

THEOREM 12. *The biorthogonal Chebyshev–Markov system for a given function* $\sigma(x)$ *is unique; the polynomial* $Q_k(x)$ *appearing in the expression for* $\psi_k(x)$ *is the polynomial of degree* k *with first coefficient* 1 *for which*

$$\int_{-1}^{1} |Q_k(x)|\, d\sigma(x) = \min_{P_k(x)} \int_{-1}^{1} |P_k(x)|\, d\sigma(x).$$

The system of polynomials $\varphi_k(x)$ is determined after this by means of a process similar to the Schmidt orthogonalization process (the metric of space $L^1_{\sigma(x)}$ is the natural one when considering series in the Chebyshev–Markov system).

§ 3. Orthogonal systems of polynomials

Various systems of orthogonal polynomials were introduced by Legendre, Jacobi, Chebyshev and other mathematicians The first of these historically was the system of Legendre polynomials. Chebyshev's studies laid the foundations of the general theory of systems of orthogonal polynomials.

Let the system

$$P_0(x),\ P_1(x),\ \ldots,\ P_n(x),\ \ldots, \tag{4.48}$$

where $P_n(x)$ is a polynomial of degree n, be orthogonal on (a, b) (in particular, the interval may be infinite) in the sense that

$$(P_m, P_n) = \int_a^b P_m(x)P_n(x)\,d\sigma(x), \tag{4.49}$$

where $\sigma(x)$ has an infinite number of growth points (in particular $d\sigma(x) = p(x)dx$, where $p(x)$ is integrable in (a, b)). Apart from a constant factor in the polynomials, the system (4.48) can be obtained by means of orthogonalization (see § 2, sec. 3) of the system of functions: 1, $x_1, x_2, \ldots x^n, \ldots$ Let us agree to write $\bar{P}_n(x)$ in future for polynomials of orthogonal system (4.48) defined by the condition that their first coefficient is equal to 1:

$$\left.\begin{aligned} &\bar{P}_0(x) = 1, \quad \bar{P}_1(x) = x + a_{10}, \ \ldots \\ &\ldots, \ \bar{P}_n(x) = x^n + a_{n,\,n-1}x^{n-1} + \ldots + a_{n,\,0}, \end{aligned}\right\} \tag{4.50}$$

and $\hat{P}_n(x)$ for polynomials of the orthonormal system, i.e.

$$\int_a^b \hat{P}_n^2(x)\,d\sigma(x) = 1. \tag{4.51}$$

The Fourier coefficients of $f(x)$ in an expansion in polynomials $\{\hat{P}_n(x)\}$ have the form

$$c_n = \int_a^b f(x)\hat{P}_n(x)\,d\sigma(x).$$

In particular, if $d\sigma(x) = G(x)\,dx$, we have

$$c_n = \int_a^b \varphi(x)\hat{P}_n(x)\,dx, \qquad \varphi(x) = f(x)G(x). \tag{4.52}$$

Notice, that, to evaluate integral (4.52), it is not necessary to find the values of $\hat{P}_n(x)$ throughout the interval. On writing

$$\varphi^{(-1)}(x) = \int_a^x \varphi(\xi)\,d\xi \quad \text{and for} \quad i > s \ \ \varphi^{(-1)}(x) = \int_a^x \varphi^{(-i+1)}(\xi)\,d\xi,$$

we obtain, by integration by parts,

$$c_n = \varphi^{(-1)}(b)\hat{P}_n(b) - \varphi^{(-2)}(b)\hat{P}_n'(b) + \ldots + (-1)^n\varphi^{(-n-1)}(b)\hat{P}_n^{(n)}(b).$$

Therefore, to find the coefficients $c_0, c_1, \ldots, c_m$, we only need to find, by successive integration of $\varphi(x)$, the values of $\varphi^{(-1)}(b)$, $\varphi^{(-2)}(b)$, ..., $\varphi^{(-m)}(b)$, the values of the polynomials $\hat{P}_k(x)$ ($k = 1, 2, \ldots, m$), and their m derivatives (for $k = 0, 1, \ldots, m$) at the point b.

1. Zeros of orthogonal polynomials

THEOREM 13. *The n-th degree polynomial* $P_n(x)$ *of the orthogonal system* (4.48) *has n real distinct simple roots, all of which lie in* (a, b).

THEOREM 14. *The zeros of the polynomials* $P_n(x)$ *and* $P_{n-1}(x)$ *of system* (4.48) *alternate: between any two zeros of* $P_n(x)$ *there is a zero of* $P_{n-1}(x)$.

It follows from Theorem 14 that $P_n(x)$ and $P_{n-1}(x)$ have no common zeros.

THEOREM 15. *If* $\lambda(x)$ *is the number of changes of sign in the series* $P_0(x), P_1(x), \ldots, P_n(x)$, *the number of roots of the polynomial* $P_n(x)$ *in the interval* (α, β) *is equal to the difference* $\lambda(\alpha)-\lambda(\beta)$ *(the property of so-called Sturm systems).*

2. Recurrence relations for orthogonal polynomials

Let $\bar{P}_{n+2}(x)$, $\bar{P}_{n+1}(x)$, $\bar{P}_n(x)$ be any three successive polynomials of system (4.48), satisfying condition (4.50). In this case the following relationship holds:

$$\bar{P}_{n+2}(x) = (x-\alpha_{n+2})\bar{P}_{n+1}(x) - \lambda_{n+1}\bar{P}_n(x), \tag{4.53}$$

where the parameters α_{n+2} and λ_{n+1} are given by:

$$\left.\begin{aligned} \alpha_{n+2} &= \frac{\int_a^b x\bar{P}_{n+1}^2(x)\,d\sigma(x)}{\int_a^b \bar{P}_{n+1}^2(x)\,d\sigma(x)}; \\ \lambda_{n+1} &= \frac{\int_a^b \bar{P}_{n+1}^2(x)\,d\sigma(x)}{\int_a^b \bar{P}_n^2(x)\,d\sigma(x)}. \end{aligned}\right\} \tag{4.54}$$

Obviously, $\lambda_{n+1} > 0$.

The parameters α_{n+2}, λ_{n+1} are expressible in terms of the coefficients of polynomials $\overline{P}_{n+2}$, $\overline{P}_{n+1}$, $\overline{P}_n$. If

$$\overline{P}_k(x) = x^k + \sum_{j=0}^{k-1} a_{kj}x^j,$$

then

$$\alpha_{n+2} = a_{n+1,\,n} - a_{n+2,\,n+1},$$

$$\lambda_{n+1} = a_{n+1,\,n-1} - \alpha_{n+2}a_{n+1,\,n} - a_{n+2,\,n}.$$

It follows from formulae (4.54) that, if (a, b) is a finite interval, we have for all n:

where

$$\left.\begin{aligned} a < \alpha_{n+2} &< b, \\ 0 < \lambda_{n+1} &< c^2, \\ \text{where} \quad c^2 = \max&\,(|a|,\ |b|). \end{aligned}\right\} \tag{4.55}$$

For examples of such recurrence relations see § 4, sec. 5–10.

3. Power moments. The expression of orthogonal polynomials in terms of power moments

The numbers

$$\mu_n = \int_a^b x^n \, d\sigma(x) \tag{4.56}$$

are called the *power moments of the weight* $\sigma(x)$.

In the case of the system

$$1,\ x,\ x^2,\ \ldots,\ x^n,\ \ldots$$

the pair-wise inner products are expressible in terms of the power moments:

$$(x^m, x^n) = \int_a^b x^m x^n \, d\sigma(x) = \mu_{n+m}.$$

Gram's determinant of the system of powers is

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \neq 0. \tag{4.57}$$

The process of orthogonalization of a system of powers yields the expression for the polynomials $\hat{P}_n(x)$ of (4.51):

$$\hat{P}_n(x) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & x^n \end{vmatrix}, \tag{4.58}$$

and for the polynomials $\overline{P}_n(x)$ of (4.50):

$$\overline{P}_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \cdots & \mu_{n-1} & 1 \\ \mu_1 & \cdots & \mu_n & x \\ \cdot & \cdot & \cdot & \cdot \\ \mu_n & \cdots & \mu_{2n-1} & x^n \end{vmatrix}. \tag{4.59}$$

The parameters α and λ of the recurrence relation (4.53) can also be expressed in terms of the power moments:

$$\begin{aligned} {}_{n+2} = \frac{1}{\Delta_{n+1}} &\begin{vmatrix} \mu_0 & \cdots & \mu_n & \mu_{n+2} \\ \mu_1 & \cdots & \mu_{n+1} & \mu_{n+3} \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{n+1} & \cdots & \mu_{2n+1} & \mu_{2n+3} \end{vmatrix} - \\ &- \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \mu_1 & \cdots & \mu_n & \mu_{n+2} \\ \cdot & \cdot & \cdot & \cdot \\ \mu_n & \cdots & \mu_{2n-1} & \mu_{2n+1} \end{vmatrix}, \end{aligned} \tag{4.60}$$

where $n = 0, 1, 2, \ldots$;

$$\alpha_{-1} = \frac{\mu_1}{\mu_0}, \quad \lambda_{n+1} = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} \qquad (n = 0, 1, 2, \ldots),$$

while $\Delta_{-1} = 1$.

4. The connection between orthogonal polynomials and continued fractions

The fraction

$$\frac{\overline{P}_n(t) - \overline{P}_n(x)}{t - x}$$

is a symmetric polynomial of degree $n-1$ in t and x. Hence the function

$$R_{n-1}(x) = \int_a^b \frac{\bar{P}_n(t)-\bar{P}_r(x)}{t-x}\,d\sigma(t) \tag{4.61}$$

is a polynomial of degree n—1 in x.

The polynomials $R_{n+1}(x)$, $R_n(x)$ and $R_{n-1}(x)$ satisfy the same recurrence relation (4.53) as polynomials $\bar{P}_{n+2}(x)$, $\bar{P}_{n+1}(x)$, $\bar{P}_n(x)$. This fact indicates (see Chapter V) that $R_{n-1}(x)$ and $\bar{P}_n(x)$ are respectively the numerator and denominator of the nth order convergent of a Chebyshev type continued fraction:

$$\frac{\lambda_0}{x-\alpha_1}+\frac{\lambda_1}{x-\alpha_2}+\ldots+\frac{\lambda_n}{x-\alpha_{n+1}}\,. \tag{4.62}$$

The polynomials $R_{n-1}(x)$ are described as polynomials of the *second kind* with respect to the weight $\sigma(x)$. They are the denominators of the continued fraction

$$\frac{\lambda_1}{x-\alpha_2}+\frac{\lambda_2}{x-\alpha_3}+\ldots+\frac{\lambda_n}{x-\alpha_{n+1}}+\ldots \tag{4.63}$$

and are orthogonal with respect to some other weight.

If the interval (a, b) is finite, the sequence $R_{n-1}(x)/\bar{P}_n(x)$ of convergents of the continued fraction (4.62) is convergent for all x lying outside (a, b) to the value of the integral

$$\int_a^b \frac{d\sigma(t)}{x-t}\,. \tag{4.64}$$

When the interval (a, b) is infinite, the continued fraction (4.62) is not always convergent. The question of its convergence is related to the determination of the corresponding problem of moments.

Regardless of the convergence, the continued fraction (4.62) is related to the integral (4.64) by the following property: the expansion of the convergent $R_{n-1}(x)/\bar{P}_n(x)$ of (4 62) as a series in negative powers of x coincides as far as terms of the order $1/x^{2n}$ (inclusive) with the series

$$\frac{\mu_0}{x}+\frac{\mu_1}{x^2}+\frac{\mu_2}{x^3}+\ldots+\frac{\mu_{2n-1}}{x^{2n}}+\frac{\mu_{2n}}{x^{2n+1}}+\ldots, \tag{4.65}$$

which can be obtained if the function $1/(x-t)$ in the integral (4.63) is expanded as a series in powers of $1/x$ and formally integrated term by term (the convergence of series (4.65) is of no consequence).

A continued fraction of the Chebyshev type, possessing the above property, is described as *associated* (assoziierte, Perron, ref. 11) with the series (4.65).

In the case $\sigma(x) \equiv$ const, the continued fraction (4.62) can be transformed, for $x < 0$, into a *continued fraction of the Stieltjes type*

$$\frac{1}{a_1x} + \frac{1}{a_2} + \frac{1}{a_3x} + \frac{1}{a_4} + \dots \tag{4.66}$$

such that the nth convergent of (4.62) coincides with the $2n$th convergent of (4.66). In view of this, the expansion as a series in negative powers of x of the $2n$th convergent of the fraction (4.66) coincides with series (4.65) up to and including the terms $1/x^{2n}$. A fraction of the Stieltjes type, possessing this property with respect to the series (4.65), is described as *corresponding to it* (correspondierende, Perron, ref. 11). We know from the theory of continued fractions that every continued fraction of the Stieltjes type (with $a_\nu \neq 0$) has a corresponding series, and every continued fraction of the Chebyshev type has an associated series.

The converse problem is of great importance in analysis: transformation of a previously assigned series in negative powers of x into a corresponding or associated continued fraction, yielding rational approximations for the function given by the series.

Not every previously assigned series has a corresponding or an associated continued fraction. A series possessing this property is described as *seminormal.*

THEOREM 16. *The series*

$$\frac{c_0}{x} + \frac{c_1}{x^2} + \dots + \frac{c_n}{x^n+1} + \dots$$

is seminormal if and only if all the determinants

$$\varphi_n = \begin{vmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_1 & c_2 & \dots & c_n \\ \dots & \dots & \dots & \dots \\ c_{n-1} & c_n & \dots & c_{2n-2} \end{vmatrix} \quad \textit{and} \quad \psi_n = \begin{vmatrix} c_1 & c_2 & \dots & c_{n-1} \\ c_2 & c_3 & \dots & c_n \\ \dots & \dots & \dots & \dots \\ c_{n-1} & c_n & \dots & c_{2n-3} \end{vmatrix}$$

are non-zero.

In this case, the coefficients of continued fraction (4.66) corresponding to this series are expressed in terms of φ_n and ψ_n as follows

$$a_1 = \varphi_1, \quad a_{2\nu} = -\frac{\psi_{\nu+1}\varphi_{\nu-1}}{\varphi_\nu\psi_\nu}, \quad a_{2\nu+1} = -\frac{\psi_{\nu+1}\varphi_\nu}{\varphi_{\nu+1}\psi_\nu},$$

where $\varphi_0 = 1$, $\psi_1 = 1$.

The fraction (4.66) corresponding to the series can be transformed by a contraction (see Chapter V, § 1, sec. 3) into an associated fraction.

5. The conversion of orthogonal expansions into a sequence of approximating fractions

A problem can be posed similar to that of converting the series

$$\frac{c_0}{x}+\frac{c_1}{x^2}+\frac{c_2}{x^3}+\ldots+\frac{c_n}{x^{n+1}}+\ldots$$

into a continued fraction

$$\frac{1}{a_1x}+\frac{1}{a_2}+\frac{1}{a_3x}+\ldots,$$

namely, conversion of the series

$$R(x) = \sum_{k=0}^{\infty} c_k\omega_k(x), \tag{4.67}$$

where $\{\omega_k(x)\}$ is a system of functions orthogonal with respect to some weight in (a, b), into a sequence of rational fractions of the form

$$\frac{P_n(x)}{q_n(x)} = \frac{\sum\limits_{i=0}^{n-\left[\frac{n}{2}\right]} a_i\omega_i(x)}{\sum\limits_{i=0}^{\left[\frac{n}{2}\right]} b_i\omega_i(x)}. \tag{4.68}$$

Here, in analogy with the conversion of power series into continued fractions the numbers a_i and b_i are chosen so that the first $n+1$ terms of the expansion of the fraction (4.68) in an orthogonal system $\{\omega_i(x)\}$ coincide with the same terms in the series (4.67). The fraction (4.68) will also be termed in this case the *n-th convergent*.

The problem of finding such a convergent is not solvable in the general case. However, the problem has a solution for certain systems namely those satisfying the relationship

$$\omega_k(x)\omega_1(x) = A_k\omega_{k+1}(x) + B_k\omega_k(x) + C_k\omega_{k-1}(x). \qquad (4.69)$$

Numerous orthogonal systems satisfy (4.69), including in particular all orthogonal systems of polynomials, the system of trigonometric functions $\{\cos nx\}$, of Bessel functions $\{J_n(x)\}$, and so on.

The formal process of converting the series (4.67) into the fraction (4.68) can be accomplished for these systems as follows. The fraction (4.68) is sought as the nth convergent of a continued fraction of the form

$$\frac{\alpha_0}{1} + \frac{\omega_1(x)+\alpha_1}{\beta_1\omega_1(x)+\gamma_1} + \frac{\omega_1(x)+\alpha_2}{\gamma_2} + \frac{\omega_1(x)+\alpha_3}{\beta_2\omega_1(x)+\gamma_3} + \dots \qquad (4.70)$$

The requirement that the series (4.67) and the fraction (4.68) correspond can now be written as

$$P_n(x) - q_n(x)\sum_{k=0}^{\infty} c_k\omega_k(x) = d_{n+1}^{(n)}\omega_{n+1}(x) + d_{n+2}^{(n)}\omega_{n+2}(x) + \dots \qquad (4.71)$$

Comparison of the coefficients of like functions $\omega_0(x), \dots, \omega_n(x)$ in (4.71) leads to the following expressions for the coefficients α_i, β_i, γ_i $(i = 1, \dots, n)$ of continued fraction (4.70):

$$\left.\begin{aligned}
&\alpha_1 = 1, \quad \beta_1 = \gamma_1 = 0, \quad \gamma_2 = B_1, \quad \gamma_n = \frac{d_{n-1}^{(n-2)}C_{n-1}}{d_{n-2}^{(n-3)}},\\
&\beta_n = \frac{d_{n-1}^{(n-2)}B_{n-1} + d_n^{(n-2)}C_n + \gamma_n d_{n-1}^{(n-3)}}{d_{n-1}^{(n-2)}},\\
&\alpha_n = -\frac{d_{n+1}^{(n-2)}C_{n+1} + d_n^{(n-2)}B_n + d_{n-1}^{(n-2)}A_{n-1} + \beta_n d_n^{(n-2)} + \gamma_n d_n^{(n-3)}}{d_n^{(n-1)}};
\end{aligned}\right\} \qquad (4.72)$$

using these, we can find $P_n(x)$ and $q_n(x)$ from the recurrence relations

$$\left.\begin{aligned}
P_n(x) &= \alpha_n P_{n-1}(x) + [\omega_1(x)+\beta_n]P_{n-2}(x) + \gamma_n P_{n-3}(x),\\
q_n(x) &= \alpha_n q_{n-1}(x) + [\omega_1(x)+\beta_n]q_{n-2}(x) + \gamma_n q_{n-3}(x),
\end{aligned}\right\} \qquad (4.73)$$

where we have put

$$\begin{aligned} &q_{-2} = 0, \quad q_{-1} = 0, \quad q_0 = 1, \quad q_1(x) = 1, \\ &P_{-2} = 0, \quad P_{-1} = C_1, \quad P_0 = C_0, \quad P_1(x) = C_0 + C_1\omega_1(x) \end{aligned}$$

(here, A_n, B_n, C_n are the coefficients of relationship (4.69)).

To obtain rational approximations of sufficiently rapid convergence by this method, an effective approach is often to employ expansions in an orthogonal system of Chebyshev polynomials of the *first kind* (see § 4, sec. 7).

6. Orthogonal polynomials and quadrature formulae of the Gaussian type

The formula

$$\int_a^b f(x)\, d\sigma(x) \approx \sum_{i=1}^{n} A_i^{(n)} f(x_i^{(n)}) \tag{4.74}$$

for approximate evaluation of the definite integral is called a *quadrature formula of the Gaussian type* if the *base-points* $x_i^{(n)}$ $(i = 1, \ldots, n)$ and coefficients $A_i^{(n)}$ $(i = 1, 2, \ldots, n)$ are chosen so that (4.74) is accurate when $f(x)$ is any polynomial of degree not exceeding $2n-1$.

THEOREM 17. *If* (4.74) *is a quadrature formula of the Gaussian type, accurate for polynomials of degree not exceeding* $2n-1$, *its base-points*

$$x_1^{(n)}, \quad x_2^{(n)}, \quad \ldots, \quad x_n^{(n)}$$

are the n roots of the polynomial $P_n(x)$ *of the system of polynomials orthogonal in* (a, b) *with respect to the weight* $\sigma(x)$, *while the coefficients* $A_i^{(n)}$ *are the coefficients in the expansion of the n-th convergent* $R_{n-1}(x)/\bar{P}_n(x)$ *of the continued fraction* (4.62) *into a sum of simple fractions*

$$\frac{R_{n-1}(x)}{\bar{P}_n(x)} = \sum_{i=1}^{n} A_i^{(n)} \frac{1}{x + x_i^{(n)}},$$

i.e.

$$A_i^{(n)} = \frac{R_n - 1(x_i^{(n)})}{\bar{P}_n'(x_i^{(n)})}. \tag{4.75}$$

Alternatively, (4.75) *can be written as*

$$A_i^{(n)} = \int_a^b \frac{\bar{P}_n(t)\, d\sigma(t)}{(t - x_i^{(n)})\bar{P}_n'(t)}.$$

In the particular case $d\sigma(x) = dx$ and $[a, b] = [0, 1]$, (4.74) is the ordinary Gaussian quadrature formula, the base-points of which are the roots of the Legendre polynomial of degree n.

7. The closure of an orthogonal system of polynomials

The necessary and sufficient condition for closure of a system of polynomials $\{P_n(x)\}$, orthogonal in (a, b) with respect to the weight $\sigma(x)$ in the space $L^2_{\sigma(x)}(a, b)$, is that the problem of moments for the sequence of moments of the weight $\sigma(x)$ be determinate or that $\sigma(x)$ be its extremal solution (see ref. 7).

The problem of moments for a finite segment $[a, b]$ is always determinate, so that orthogonal systems of polynomials with respect to any weight on a finite segment are closed. In particular, the systems of Legendre, Chebyshev and Jacobi polynomials are closed (see § 4).

The problems of moments are determinate for the Laguerre weight (see § 4, sec. 9) on $(0, +\infty)$, and for the Hermite weight (see § 4, sec. 4) on $(-\infty, +\infty)$, i.e. these systems of polynomials are closed.

Hence the Fourier series in orthogonal polynomials of a function $f(x)$ of $L^2_{\sigma(x)}$ are convergent in the mean to this function in the case of all the classical weights.

A discussion of the convergence of series in orthogonal polynomials $\{P_n(x)\}$ at every point and of the uniform convergence requires asymptotic estimates of $|P_n(x)|$ as $n \to \infty$.

8. Christoffel's formula. The convergence of Fourier series in orthogonal polynomials

Let $\{\bar{P}_n(x)\}$ be an orthonormal system of polynomials in (a, b) with respect to the weight $\sigma(x)$, and let $f(x)$ be any function of the space $L^2_{\sigma(x)}(a, b)$.

Let us write s_n for the nth partial sum of the Fourier series of $f(x)$ with respect to the system $\{\hat{P}_n(x)\}$:

$$s_n(x) = \sum_{k=0}^{n} C_k \hat{P}_k(x). \tag{4.76}$$

If we make use of the expressions for the Fourier coefficients, $s_n(x)$ can be written as

$$s_n(x) = \int_a^b f(t) \left(\sum_{k=0}^{n} \hat{P}_k(t) \hat{P}_k(x) \right) d\sigma(t). \tag{4.77}$$

The expression

$$K_n(t, x) = \sum_{k=0}^{n} \hat{P}_k(t) \hat{P}_k(x) \tag{4.78}$$

is called the *kernel of the integral* (4.77). The following formula holds for the kernel:

$$K_n(t, x) = \sqrt{\lambda_{n+1}} \frac{\hat{P}_{n+1}(t)\hat{P}_n(x) - \hat{P}_{n+1}(x)\hat{P}_n(t)}{t-x}, \tag{4.79}$$

where λ_{n+1} is given by (4.54). Formula (4.79), obtained by *Christoffel* for the case $a = -1$, $b = +1$, $d\sigma(x) = dx$, and generalized by *Darboux* to the case of any weight, is known as the *Christoffel-Darboux formula.*

Formula (4.77) and the relationship

$$\int_a^b K_n(t, x)\, d\sigma(x) = 1$$

lead to a formula for the remainder of the Fourier series of $f(x)$ in the system $\{\hat{P}_n(x)\}$:

$$s_n(x) - f(x) = \sqrt{\lambda_{n+1}} \int_a^b \varphi_x(t) [\hat{P}_{n+1}(t)\hat{P}_n(x) - \hat{P}_{n+1}(x)\hat{P}_n(t)]\, d\sigma(t), \tag{4.80}$$

where

$$\varphi_x(t) = \frac{f(t) - f(x)}{t-x}$$

The next two theorems, on the convergence of the Fourier series of $f(x)$, follow from (4.80).

THEOREM 18. *If all the polynomials* $P_n(x)$ *are bounded at the point* x *and the function* $\varphi_x(t)$ *belongs to* $L^2_{\sigma(t)}$ (a, b), *the Fourier series*

$$\sum_{k=0}^{\infty} C_k\hat{P}_k(x)$$

is convergent at x *and*

$$f(x) = \sum_{k=0}^{\infty} C_k\hat{P}_k(x).$$

DEFINITION. *The sequence of functions* $\varphi_1(x)$, $\varphi_2(x)$, $\ldots$, $\varphi_n(x)$, $\ldots$ *is said to be uniformly bounded if there exists a constant* M *such that* $|\varphi_n(x)| < M$ *for all the functions of the system.*

THEOREM 19. *If the polynomials* $P_n(x)$ *are uniformly bounded on* (a, b), *given any function* $f(x)$, *integrable on* (a, b) *with respect to the weight* $\sigma(x)$, *its Fourier series in the system* $\{P_n(x)\}$ *is convergent to the value* $f(x)$ *at every point* x *for which*

$$\int_a^b \varphi_x(t)\,d\sigma(t)$$

exists.

The following theorems establish the convergence of the Fourier series of $f(x)$ in an orthogonal system of polynomials $\{P_n(x)\}$, having regard to the structural properties of $f(x)$ instead of the properties of the system.

THEOREM 20. *If* $f(x)$ *satisfies the Lipschitz condition with index* $\alpha > \frac{1}{2}$,

$$|f(x_2)-f(x_1)| < M\,|x_2-x_1|^{\alpha},$$

then the Fourier series of $f(x)$ *in a system of orthogonal polynomials* $\{P_n(x)\}$ *is convergent almost everywhere on* (a, b).

DEFINITION. *The function* $L_n(x) = \int_a^b |K_n(t, x)|\,d\sigma(t)$ *is known as the Lebesque function of the orthonormal system* $\{\hat{P}_n(x)\}$.

THEOREM 21. *Let* $f(x)$ *be continuous and let its best approximation by polynomials of degree* n

$$E_n(f) = \inf_{Q_n(x)} \max_{(a,\,b)} |f(x)-Q_n(x)|$$

satisfy the relationship

$$L_n(x_0)\cdot E_n(f) \to 0 \qquad \text{as } n \to \infty.$$

Then the Fourier series of $f(x)$ *in the orthogonal system* $\{\hat{P}_n(x)\}$ *is convergent, at* $x = x_0$, *to the value* $f(x_0)$.

§ 4. Classical systems of orthogonal polynomials

1. Pearson's differential equation

The equation

$$\frac{\varrho'}{\varrho} = \frac{\alpha_0+\alpha_1 x}{\beta_0+\beta_1 x+\beta_2 x^2}\left(=\frac{\alpha(x)}{\beta(x)}\right) \tag{4.81}$$

is known as *Pearson's equation.* Solutions of the equation are called *Pearson functions.* The equation was introduced by Pearson for representing empirical laws of distribution. When $\alpha_1 = -1$, $\alpha_0 = 0$, $\beta_0 = 1$, $\beta_1 = \beta_2 = 0$, the solution is the classical function $e^{-\frac{1}{2}x^2}$ — *the density of a normal distribution law.*

The weights of all the most important systems of orthogonal polynomials are Pearson functions.

Jacobi's weight is

$$\varrho(x) = (1-x)^{\lambda}(1+x)^{\mu}, \tag{4.82}$$

where $\lambda > -1$, $\mu > -1$; $\varrho(x)$ is defined on $[-1, 1]$ and all its moments exist there. Pearson's equation for Jacobi's weight is

$$\frac{\varrho'}{\varrho} = \frac{(\mu-\lambda)-(\mu+\lambda)x}{1-x^2}. \tag{4.83}$$

Chebyshev's weight is a particular case of that of Jacobi, corresponding to $\lambda = -\frac{1}{2}$, $\mu = -\frac{1}{2}$;

$$\varrho(x) = \frac{1}{\sqrt{1-x^2}}. \tag{4.84}$$

Pearson's differential equation for Chebyshev's weight is

$$\frac{\varrho'}{\varrho} = \frac{x}{1-x^2}. \tag{4.85}$$

Legendre's weight $\varrho(x) \equiv 1$ is a particular case of that of Jacobi, corresponding to $\lambda = 0$, $\mu = 0$. Pearson's equation for it is

$$\frac{\varrho'}{\varrho} = \frac{1}{1-x^2}. \tag{4.86}$$

Any Pearson function, having all moments, and for which the denominator $\beta(x)$ in (4.81) has real distinct roots, is reducible to any of the above functions by means of a linear change of the independent variable.

When the denominator $\beta(x)$ has multiple or complex roots, the Pearson functions may not be weights of an orthogonal system of polynomials, since not all moments exist on the interval on which they are defined.

The Chebyshev–Laguerre weight:

$$\varrho(x) = x^{\lambda}e^{-\mu x}, \quad \text{where} \quad \lambda > -1,\ \mu > 0, \tag{4.87}$$

is defined on $(0, +\infty)$. Pearson's equation for the Chebyshev–Laguerre weight is

$$\frac{\varrho'}{\varrho} = \frac{\lambda-\mu x}{x}. \tag{4.88}$$

Any Pearson function, having all moments on $(0, +\infty)$, for which the denominator $\beta(x)$ in (4.81) is a polynomial of the first degree ($\beta_2 = 0$), is reducible to this function by means of a linear change of the independent variable.

The Chebyshev–Hermite weight:

$$\varrho(x) = e^{-x^2} \tag{4.89}$$

is defined on $(-\infty, +\infty)$. Pearson's equation becomes

$$\frac{\varrho'}{\varrho} = -2x.$$

Any Pearson function, having all moments on $(-\infty, +\infty)$, for which $\beta(x) = \text{const.}$ in (4.81) ($\beta_1 = \beta_2 = 0$), is reducible to a Chebyshev-Hermite function.

Thus all the Pearson functions that can serve as weights of an orthogonal system of polynomials are reducible to one of the above basic weights.

2. The differential equations for corresponding classes of orthogonal polynomials

The polynomials of orthogonal systems whose weights are Pearson functions satisfy linear differential equations of the second order, to which various physical problems often reduce — a fact that ensures their importance in applied mathematics.

If the weight $\varrho(x)$ of an orthogonal system of polynomials satisfies

$$\frac{\varrho'}{\varrho} = \frac{\alpha_0+\alpha_1 x}{\beta_0+\beta_1 x+\beta_2 x^2},$$

then the nth degree polynomial of this system satisfies the differential equation

$$\beta(x)y''+[\alpha(x)+\beta'(x)]y'-\gamma_n y = 0, \tag{4.90}$$

where $\gamma_n = n[\alpha_1+(n+1)\beta_2]$, and $\alpha(x)$, $\beta(x)$ are as in (4.81).

EXAMPLE 15. For Chebyshev's weight,

$$\alpha(x) = x, \qquad \alpha_1 = 1,$$
$$\beta(x) = 1-x^2, \quad \beta_2 = -1.$$

The equation for Chebyshev polynomials is of the form

$$(1-x^2)y''-xy'+n^2y = 0.$$

3. The expression, by means of the weight, of a polynomial of the *n*th degree belonging to an orthogonal system of polynomials

Let us consider an orthogonal polynomial system with Pearson's weight $\varrho(x)$ from (4.81). Let $P_n(x)$ be the nth degree polynomial of the system, orthogonal with respect to the weight function $\varrho(x)$. We can write $P_n(x)$ in the form

$$P_n(x) = A_n \frac{1}{\varrho(x)} \frac{d^n}{dx^n}\{\varrho(x)\beta^n(x)\}. \tag{4.91}$$

The formula was obtained by *Rodrigues* for Legendre polynomials (in 1814), while similar formulae were later obtained for other polynomials. Rodrigues' formula was first published in the general form (4.91) in ref. 5.

If $A_n = 1$, the coefficient of the highest term in $P_n(x)$ is

$$\tilde{a}_n = \prod_{k=n+1}^{2n} (\alpha_1 + k\beta_2), \tag{4.92}$$

while for x^{n-1} it is

$$\tilde{b}_n = \frac{\alpha_0 + n\beta_1}{\alpha_1 + 2n\beta_2} n\tilde{a}_n. \tag{4.93}$$

The coefficient of the highest term in the normalized polynomial is

$$a_n^0 = \sqrt{\frac{(-1)^n \tilde{a}_n}{n! \int_a^b \varrho(x)\beta^n(x)\,dx}} \tag{4.94}$$

4. The generating function of an orthogonal system of polynomials with Pearson's weight

Let us take an orthogonal polynomial system $\{P_n(x)\}$, where $P_n(x)$ is defined by (4.91) with constant $A_n = 1$.

The *generating function* of the system is the function $\psi(z, w)$ of two complex variables z and w such that

$$\psi(z, w) = \sum_{n=0}^{\infty} \frac{P_n(z)}{n!} w^n. \tag{4.95}$$

THEOREM 22. *Given any orthogonal system of polynomials* $\{P_n(x)\}$ *with weight function* $\varrho(x)$, *satisfying condition* (4.81), *there exists a generating function* (4.95), *which is given by*

$$\psi(z, w) = \frac{1}{\varrho(z)} \frac{\varrho(\xi w)}{1 - w\beta'(\xi w)}, \tag{4.96}$$

where ξ_w *is the root of the quadratic equation*

$$\xi - z - w\beta(\xi) = 0 \tag{4.97}$$

that is close to z *for small* w.

Example 16. Let us find the generating function for the Legendre polynomials.

Equation (4.97) becomes in this case

$$w\xi^2 + \xi - (z+w) = 0,$$

where

$$\xi_w = \frac{1}{2w}\left(-1 \pm \sqrt{1+4wz+4w^2}\right)$$

is chosen in such a way that ξ_w is close to z for small w, i.e.

$$\xi_w = \frac{1}{2w}\left(-1 + \sqrt{1+4wz+4w^2}\right).$$

On applying formula (4.96) and putting $\varrho(x) \equiv 1$, we get

$$\psi(z, w) = \frac{1}{\sqrt{1+4wz+4w^2}}. \tag{4.98}$$

The derivatives of polynomials orthogonal with respect to Pearson's weight are also orthogonal polynomials with respect to the weight

$$\varrho_1(x) = \exp\left\{\int \frac{\alpha(x)+\beta'(x)}{\beta(x)}\,dx\right\}.$$

5. Legendre polynomials

The first system of orthogonal polynomials was historically the system of polynomials with the weight function $\varrho(x) \equiv 1$ on $[-1, +1]$, introduced by Legendre in 1785.

We introduce the following notation: let $L_n(x)$ be the Legendre polynomial, in which we have not fixed the factor, to an accuracy of which the system of orthogonal polynomials is defined, let $\bar{L}_n(x)$ be the nth degree polynomial with highest coefficient equal to 1, and let $\hat{L}_n(x)$ be the normalized Legendre polynomial.

Rodrigues' formula is

$$L_n(x) = A_n \frac{d^n[(x^2-1)^n]}{dx^n}. \tag{4.99}$$

Formulae (4.92)–(4.94) lead to the expressions for $\bar{L}_n(x)$ and $\hat{L}_n(x$:)

$$\bar{L}_n(x) = \frac{n!}{(2n)!}\frac{d^n}{dx^n}(x^2-1)^n, \tag{4.100}$$

$$\hat{L}_n(x) = \sqrt{\frac{2n+1}{2}}\frac{1}{(2n)!!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{4.101}$$

The explicit expression for a Legendre polynomial is

$$L_n(x) = A_n\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\frac{(2n-2k)!}{(n-2k)!}C_n^k x^{n-2k}. \tag{4.102}$$

It follows from (4.102) that $L_n(x)$ is an even function in x when n is even, and odd when n is odd.

A recurrence formula. For the Legendre polynomials, we have in (4.54):

$$\alpha_{n+2} = 0, \quad \lambda_{n+1} = \frac{(n+1)^2}{(2n+1)(2n+3)},$$

i.e.

$$\bar{L}_{n+2}(x) = x\bar{L}_{n+1}(x) - \frac{(n+1)^2}{(2n+1)(2n+3)}\bar{L}_n(x). \tag{4.103}$$

Obviously, $\bar{L}_0(x) = 1$, $\bar{L}_1(x) = x$; we further obtain from (4.103):

$$\bar{L}_2(x) = \frac{1}{3}(3x^2-1),$$

$$\bar{L}_3(x) = \frac{1}{5}(5x^3-3x),$$

$$\bar{L}_4(x) = \frac{1}{35}(35x^4-30x^2+3),$$

$$\bar{L}_5(x) = \frac{1}{315}(315x^5-350x^3+75x) \quad \text{and so on.}$$

The continued fraction (4.62) for a Legendre polynomial becomes

$$\frac{2}{x} - \frac{\frac{1}{3}}{x} - \frac{\frac{4}{15}}{x} - \frac{\frac{9}{35}}{x} - \dots - \frac{\frac{(n+1)^2}{(2n+1)(2n+3)}}{x} - \dots. \tag{4.104}$$

The denominator of the nth convergent of (4.104) is $L_n\ (x)$.

The continued fraction (4.104) is convergent to $\ln(x+1)/(x-1)$ at every point x lying outside $[-1, +1]$.

The generating function. Let $H(x, w)$ be the generating function of the Legendre polynomials (4.95); (4.98) now gives

$$H(x, w) = \frac{1}{\sqrt{1+4wx+4w^2}}, \tag{4.105}$$

and

$$\frac{1}{\sqrt{1+4wx+4w^2}} = \sum_{n=0}^{\infty} \frac{\tilde{L}_n(x)}{n!} w^n,$$

where

$$\tilde{L}_n(x) = \frac{d^n}{dx^n}(x^2-1)^n,$$

or

$$H(x, w) = 1-2xw+2(3x^2-1)w^2-4(5x^3-3x)w^3+ \\ +2(35x^4-30x^2+3)w^4+ \dots .$$

The generating function of the Legendre polynomials is often written in the form

$$H(x, t) = \frac{1}{\sqrt{1-2xt+t^2}}, \quad \text{where} \quad t = -2w,$$

so that

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{k=0}^{\infty} l_k(x)t^k,$$

where $l_k(x)$ is given by (4.99) with $A_k = 1/(2k)!!$.

The polynomials $\bar{L}_n(x)$ and $\hat{L}_n(x)$ are expressed as follows in terms of $l_n(x)$:

$$\left.\begin{aligned} \bar{L}_n(x) &= \frac{n!}{(2n-1)!!} l_n(x); \\ \hat{L}_n(x) &= \sqrt{\frac{2n+1}{2}}\, l_n(x). \end{aligned}\right\} \tag{4.106}$$

The polynomials $l_n(x)$ satisfy the recurrence formula

$$(n+2)l_{n+2}(x) = (2n+3)xl_{n+1}(x)-(n+1)l_n(x). \tag{4.107}$$

The *differential equation* for Legendre polynomials is

$$(1-x^2)y''-2xy'+n(n+1)y = 0.$$

The *integral form* of Legendre polynomials. The polynomial

$$l_n(x) = \frac{1}{(2n)!!}\frac{d^n}{dx^n}(x^2-1)^n$$

can be written in the form

$$l_n(x) = \frac{1}{\pi}\int_0^{\pi}\left(x+i\sqrt{1-x^2}\cos\varphi\right)^n d\varphi \qquad (4.108)$$

The integral in (4.108) is known as a *Laplace integral* (it has a real value for real x, in spite of the integrand being complex). The polynomials $l_n(x)$ are uniformly bounded on the segment of orthogonality $[-1, +1]$.

It follows from (4.108) that

$$|l_n(x)| \leqq 1 \qquad (4.109)$$

for

$$|x| \leqq 1.$$

A stricter inequality holds for points lying inside the interval $(-1, +1)$:

$$|l_n(x)| \leqq \sqrt{\frac{\pi}{2n}}\,\frac{1}{\sqrt{1-x^2}}. \qquad (4.110)$$

Turan's inequality is

$$l_n^2(x)-l_{n-1}(x)l_{n+1}(x) \geqq 0$$

for $n \geqq 1$, $-1 \leqq x \leqq 1$.

The following are expansions of certain functions in Legendre polynomials:

$$x^{2n} = \frac{1}{2n+1}l_0(x)+$$

$$+\sum_{k=1}^{n}(4k+1)\frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)}l_{2k}(x),$$

$$x^{2n+1} = \frac{3}{2n+3} l_1(x) + $$
$$+ \sum_{k=1}^{n} (4k+3) \frac{2n(2n-2)\ldots(2n-2k+2)}{(2n+3)(2n+5)\ldots(2n+2k+3)} l_{2k+1}(x),$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+1) \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 l_{2k}(x), \quad |x| < 1,$$

$$\frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+3) \frac{(2k-1)!!\,(2k+1)!!}{2^{2k+1} k!\,(k+1)!} l_{2k+1}(x), \quad |x| < 1.$$

$$\sqrt{1-x^2} = \frac{\pi}{2} \left\{ \frac{1}{2} - \sum_{k=1}^{\infty} (4k+1) \frac{(2k-3)!!\,(2k-1)!!}{2^{2k+1} k!\,(k+1)!} l_{2k}(x) \right\},$$
$$|x| < 1,$$

$$\arcsin x = \frac{\pi}{2} \sum_{k=0}^{\infty} \left(\frac{(2k-1)!!}{2^k k!} \right)^2 [l_{2k+1}(x) - l_{2k-1}(x)], \quad |x| < 1.$$

The convergence of Fourier series in Legendre polynomials.

THEOREM 23. *If $f(x)$ has a continuous second derivative in* $[-1, +1]$, *it can be expanded as a uniformly convergent series in Legendre polynomials in* $[-1, +1]$.

By (4.106) and (4.109), we have the following inequality for normalized Legendre polynomials:

$$|\hat{L}_n(x)| < \sqrt{\frac{2n+1}{2}} \quad \text{for} \quad |x| \leqq 1. \tag{4.111}$$

Formula (4.111) and the corresponding inequality for the kernel $\bar{K}_n(t, x)$ of $\hat{L}_n(x)$ give an inequality for the Lebesgue function (see § 3, sec. 8):

$$L_n(x) \leqq (n+1)^2. \tag{4.112}$$

A consequence of Theorem 21 (see § 3, sec. 8) and (4.112) is:

THEOREM 24. *Every continuous function $f(x)$, the best approximation of which satisfies the condition*

$$\lim_{n \to \infty} n^2 E_n(f) = 0 \tag{4.113}$$

can be expanded as a uniformly convergent series in $[-1, +1]$ *in Legendre polynomials.*

The following is a consequence of Theorem 23 and formulae (4.106) and (4.110):

THEOREM 25. *A function $f(x)$, the square of which is integrable in* $[-1, 1]$, *can be expanded as a Fourier series in Legendre polynomials convergent to $f(x)$ at every point x for which the integral exists:*

$$\int_{-1}^{1}\left(\frac{f(t)-f(x)}{t-x}\right)^2 dt. \tag{4.114}$$

REMARK. Condition (4.114) is satisfied, in particular, if the finite derivative $f'(x)$ exists at the point x.

THEOREM 26. *Let $f(x)$ be a function, the square of which is integrable in* $[-1, 1]$, *and suppose the left- and right-hand limits $f(x-0)$ and $f(x-0)$ exist at the point x In this case, if the integrals*

$$\int_{-1}^{x}\left(\frac{f(t)-f(x-0)}{t-x}\right)^2 dt \quad \text{and} \quad \int_{x}^{1}\left(\frac{f(t)-f(x+0)}{t-x}\right)^2 dt$$

are finite, the Fourier series in Legendre polynomials is convergent at the point x to $\frac{1}{2}[f(x-0)+f(x+0)]$

THEOREM 27. *If $f(x)$ satisfies the Dini–Lipschitz condition in* $[-1, +1]$:

$$\lim_{\delta\to 0}\omega(\delta)\ln(\delta) = 0$$

(*where* $\omega(\delta) = \sup_{|x_1-x_2|<\delta}\{|f(x_1)-(f(x_2)|\}$ *is the oscillation of* $f(x)$), *it can be expanded at every point of the interval as a Fourier series in Legendre polynomials, the convergence being uniform in every segment* $[-1+h, 1-h]$ $(h>0)$.

6. Jacobi polynomials

The *Jacobi polynomials* are polynomials orthogonal in $[-1, +1]$ with respect to the weight

$$\varrho(x) = (1-x)^{\lambda}(1+x)^{\mu}, \tag{4.115}$$

where $\lambda > -1$, $\mu > -1$. The weight function (4.115) is Pearson's weight (see § 4, sec. 1). The above definition specifies the Jacobi polynomials apart from a constant factor. If this factor is not fixed, we shall denote the polynomials by $J_n^{(\lambda,\mu)}(x)$; we write $\bar{J}_n^{(\lambda,\mu)}(x)$ if the coefficient of x^n is unity, and $\hat{J}_n^{(\lambda,\mu)}(x)$ if the polynomial is normalized.

The Legendre polynomials (see sec. 5) are a particular case of Jacobi polynomials, with $\lambda = \mu = 0$. The case $\lambda = \mu = \frac{1}{2}$ and $\lambda = \mu = -\frac{1}{2}$ are specially treated later (see sec. 7, 8). The polynomials corresponding to these values of λ and μ are called *Chebyshev polynomials* (of the *second* and *first kind* respectively).

In general, the case $\lambda = \mu$ has certain special features. The Jacobi polynomials with $\lambda = \mu$ are said to be *ultraspherical.*

Rodrigues' formula. Formula (4.91) gives for Jacobi polynomials:

$$J_n^{(\lambda,\mu)}(x) = A_n(1+x)^{-\mu}\frac{d^n}{dx^n}[(1-x)^{\lambda+n}(1+x)^{\mu+n}], \tag{4.116}$$

where we have for $\bar{J}_n^{(\lambda,\mu)}(x)$:

$$A_n = (-1)^n\frac{\Gamma(\lambda+\mu+n+1)}{\Gamma(\lambda+\mu+2n+1)}, \tag{4.117}$$

and for $\hat{J}_n^{(\lambda,\mu)}(x)$:

$$A_n = (-1)^n\sqrt{2^{-(\lambda+\mu+2n+1)}\frac{\Gamma(\lambda+\mu+n+1)\Gamma(\lambda+\mu+2n+1)}{\Gamma(\lambda+n+1)\Gamma(\mu+n+1)n!}}. \tag{4.118}$$

These formulae are suitable for $n > 0$. They are also true for $n = 0$, provided $\lambda+\mu+1 \neq 0$; if $\lambda+\mu+1 = 0$, they lose their meaning at $n = 0$, though it is clear that

$$\bar{J}_0^{(\lambda,\mu)}(x) = 1.$$

An explicit expression. If $A_n = (-1)^n/2^n n!$ in Rodrigues' formula (4.116), we shall write $j_n^{(\lambda,\mu)}(x)$ for the corresponding Jacobi polynomial. It is often convenient to discuss this Jacobi polynomial. For it, the following formula holds:

$$j_n^{(\lambda,\mu)}(x) = \\ = \frac{1}{2^n n!}\sum_{k=0}^{n} C_n^k\frac{\Gamma(\lambda+n+1)\Gamma(\mu+n+1)}{\Gamma(\lambda+n-k+1)\Gamma(\mu+k+1)}(x-1)^{n-k}(x+1)^k. \tag{4.119}$$

A recurrence relation. The general recurrence relation (4.54) for three consecutive polynomials of a system of orthogonal polynomials becomes, in the case of the system of Jacobi polynomials;

$$\bar{J}^{(\lambda,\,\mu)}_{n+2}(x) = (x-\alpha_{n+2})\,\bar{J}^{(\lambda,\,\mu)}_{n+1}(x) - \lambda_{n+1}\bar{J}^{(\lambda,\,\mu)}_{n}(x), \tag{4.120}$$

where

$$\left.\begin{aligned} \alpha_{n+2} &= \frac{\mu^2-\lambda^2}{(\lambda+\mu+2n+2)(\lambda+\mu+2n+4)}\,, \\ \lambda_{n+1} &= \frac{(\lambda+n+1)(\mu+n+1)(\lambda+\mu+n+1)}{(\lambda+\mu+2n+1)(\lambda+\mu+2n+2)^2(\lambda+\mu+2n+3)}\,4(n+1). \end{aligned}\right\} \tag{4.121}$$

Since

$$\bar{J}^{(\lambda,\,\mu)}_{0} = 1 \quad \text{and} \quad \bar{J}^{(\lambda,\,\mu)}_{1} = x - \frac{\lambda-\mu}{\lambda+\mu+2},$$

all the Jacobi polynomials can be obtained successively from (4.120). These expressions are very unwieldy, however; e.g.

$$\bar{J}^{(\lambda,\,\mu)}_{2}(x) = x^2 + 2\frac{\lambda-\mu}{\lambda+\mu+4}\,x + \frac{(\lambda-\mu)^2+(\lambda+\mu)-4}{(\lambda+\mu+3)(\lambda+\mu+4)}$$

and so on. The explicit formula (4.119) is more practical for their actual computation. Complete tables of Jacobi polynomials are available (see ref. 3). The Jacobi polynomials satisfy recurrence relations, not only with respect to the parameter n, but also with respect to the parameters λ and n, λ, μ and n, these being consequences of Gauss's formulae for the hypergeometric functions; the Jacobi polynomials are a particular case of the latter.

The generating function. Let $I(x, w)$ be the generating function of the Jacobi polynomials (see sec. 4). Formula (4.96) now gives

$$\begin{aligned} I(x, w) = \frac{2^{\lambda+\mu}}{\sqrt{1+4xw+4w^2}}\left(1+2w+\sqrt{1+4wx+4w^2}\right)^{-\lambda} \times \\ \times\left(1-2w+\sqrt{1+4wx+4w^2}\right)^{-\mu} \end{aligned} \tag{4.122}$$

and

$$I(x, w) = \sum_{n=0}^{\infty} \frac{\bar{J}^{(\lambda\mu)}_{n}(x)}{n!}\,w^n, \tag{4.123}$$

where

$$\tilde{J}_n^{(\lambda,\mu)} = (1-x)^{-\lambda}(1+x)^{-\mu}\frac{d^n}{dx^n}[(1-x)^{(\lambda+n)}(1+x)^{\mu+n}]. \tag{4.124}$$

The generating function for the Jacobi polynomials is often written in the form

$$I(x, t) = \frac{2^{\lambda+\mu}}{\sqrt{1-2xt+t^2}}(1-t+\sqrt{1-2xt+t^2})^{-\lambda}\times$$
$$\times(1+t+\sqrt{1-2xt+t^2})^{-\mu}, \tag{4.125}$$

where $t = -2w$; the coefficients of the expansion in powers of t are now the polynomials $j_n^{(\lambda,\mu)}(x)$ (see (4.119)). In the case of ultraspherical polynomials ($\lambda = \mu$), the generating function is simplified if we introduce, instead of polynomials $j_n^{(\lambda,\mu)}(x)$, the polynomials

$$\gamma_n^{(\alpha)}(x) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(n+2\alpha)}{\Gamma(2\alpha)\Gamma\left(n+\alpha+\frac{1}{2}\right)}j_n^{\lambda,\lambda}(x), \quad \text{where} \quad \alpha = \lambda+\frac{1}{2},$$

in fact,

$$(1-2xt+t^2)^{-\alpha} = \sum_{n=0}^{\infty}\gamma_n^{(\alpha)}(x)t^n. \tag{4.126}$$

By using the ultraspherical polynomials γ_n^{α}, N. Ya. Sonin obtained an analogue of Taylor's formula:

$$f(x+\alpha) = 2^{\nu}\Gamma(\nu)\sum_{n=0}^{\infty}(n+\nu)\frac{J_{n+\nu}(\alpha)}{\alpha^{\nu}}\frac{\gamma_n^{(\nu)}(iD)}{i^n}f(x), \tag{4.127}$$

where $J_k(\alpha)$ is Bessel's function, $D = d/dx$.

A differential equation for Jacobi polynomials. Differential equation (4.90) for polynomials orthogonal with respect to Pearson's weight function, becomes, in the case of Jacobi polynomials:

$$(1-x^2)y''+[(\mu-\lambda)-(\mu+\lambda+2)x]y'+n(\lambda+\mu+n+1)y = 0. \tag{4.128}$$

Inequalities for Jacobi polynomials and the convergence of the Fourier series. Given the condition

$$\sigma = \max\{\lambda, \mu\} \geqq -\frac{1}{2} \tag{4.129}$$

the following theorems hold.

THEOREM 28. *The greatest value of the modulus of* $J_n^{(\lambda,\mu)}(x)$ *is attained in the segment* $[-1, 1]$ *at one of the points* $x = \pm 1$.

THEOREM 29. *Given condition* (4.129), *the normalized Jacobi polynomials satisfy the relationship*

$$|\hat{J}_n^{(\lambda,\mu)}(x)| < Mn^{\sigma+\frac{1}{2}}$$

for all $|x| \leqq 1$.

Here M is a constant, depending on λ and μ.

It follows from Theorem 29 that the Lebesgue function of the Jacobi polynomials satisfies

$$L_n(x) < M_1 n^{2\sigma+2},$$

which leads, in conjunction with Theorem 21 (see § 3), to

THEOREM 30. Let $\sigma \geqq -\dfrac{1}{2}$ *and let* p *be a positive integer not less than* $2\sigma+2$. *Every function* $f(x)$, *defined in* $[-1, 1]$ *and having a continuous derivative of order* p, *can be expanded as a uniformly convergent Fourier series in the polynomials* $\hat{J}_n^{(\lambda,\mu)}(x)$.

7. Chebyshev polynomials of the first kind

Chebyshev polynomials of the first kind are a particular case of Jacobi polynomials (see § 4, sec. 6), corresponding to $\lambda = \mu = -\dfrac{1}{2}$. The polynomials $J_n^{\left(-\frac{1}{2},\,-\frac{1}{2}\right)}$ were first discussed by P.L. Chebyshev in 1857, when solving the problem of the best approximation of continuous functions by polynomials. We shall denote them by $T_n(x)$; these polynomials were obtained by Chebyshev in the form

$$T_n(x) = \cos(n \arccos x). \qquad (4.130)$$

Formula (4.130) defines $T_n(x)$ only on the segment $[-1, 1]$. But definition (4.130) can be extended to all values of x by means of the familiar trigonometric formula.

$$\cos n\varphi = \cos^n \varphi - C_n^2 \cos^{n-2}\varphi \sin^2\varphi + C_n^4 \cos^{n-4}\varphi \sin^4\varphi - \dots$$

(this completion of definition (4.130) will be assumed without making any special proviso in future).

The polynomials $T_n(x)$ possess many remarkable so-called *extremal properties* in addition to their general properties as polynomials of an orthogonal system; these extremal properties will be described after dealing with the general properties.

Rodrigues' formula is

$$T_n(x) = A_n \sqrt{1-x^2}\,\frac{d^n}{dx^n}\left(\frac{1}{\sqrt{1-x^2}}\right)^n. \qquad (4.131)$$

We shall write $\overline{T}_n(x)$ for the polynomial with highest coefficient equal to unity, and $\hat{T}_n(x)$ for the normalized polynomial.

An explicit expression. Formula (4.130) yields an explicit expression for the Chebyshev polynomials, where

$$\left.\begin{aligned} \overline{T}_n(x) &= \frac{1}{2^{n-1}}\cos(n \arccos x), \\ \hat{T}_n(x) &= \sqrt{\frac{2}{\pi}}\cos(n \arccos x). \end{aligned}\right\} \qquad (4.132)$$

Expression (4.130) can also be written in the form

$$\left.\begin{aligned} T_n(x) &= \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right] \\ &\text{or} \\ T_n(x) &= \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} C_{n-k}^k 2^{n-2k-1} x^{n-2k}. \end{aligned}\right\} \qquad (4.133)$$

As follows from (4.130), the zeros of $T_n(x)$ are the numbers

$$x_k^{(n)} = \cos\frac{(2k-1)\pi}{2n} \qquad (k = 1, 2, \ldots, n). \qquad (4.134)$$

The recurrence formulae

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad (4.135)$$

$$\overline{T}_n(x) = x\overline{T}_{n-1}(x) - \frac{1}{4}\overline{T}_{n-2}(x) \qquad (4.136)$$

follow from (4.120) with $\lambda = \mu = -\frac{1}{2}$. Since $T_0(x) = 1$, $T_1(x) = x$, it follows from the equation (4.135) that

$$\begin{aligned}
T_2(x) &= 2x^2-1,\\
T_3(x) &= 4x^3-3x,\\
T_4(x) &= 8x^4-8x^2+1,\\
T_5(x) &= 16x^5-20x^3+5x,\\
T_6(x) &= 32x^6-48x^4+18x^2+1
\end{aligned}$$

and so on.

In the case of Chebyshev polynomials, the continued fraction (4.62) becomes

$$\frac{\pi}{x} - \frac{\frac{1}{4}}{x} - \frac{\frac{1}{4}}{x} - \frac{\frac{1}{4}}{x} - \ldots, \tag{4.137}$$

since here,

$$\lambda_0 = \pi = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}, \quad \lambda_i = \frac{1}{4}, \quad d_i = 0.$$

The denominator of the nth convergent of the continued fraction (4.137) is $\bar{T}_n(x)$. The continued fraction is convergent, for all x outside the interval $(-1, 1)$, to the function $\pi/\sqrt{x^2-1}$.

The generating function. The coefficient of t^n in the expansion of

$$T(x, t) = \frac{1-t^2}{1-2tx+t^2} \tag{4.138}$$

in powers of t is the polynomial $T_n(x)$:

$$T(x, t) = \frac{1-t^2}{1-2tx+t^2} = T_0 + 2\sum_{n=1}^{\infty} T_n(x)t^n.$$

The differential equation for Chebyshev polynomials is

$$(1-x^2)y'' - xy' + n^2y = 0. \tag{4.139}$$

Expansion of a function as a Fourier series in Chebyshev polynomials and the comparison of this with the expansion as a Maclaurin series. Wide use is made of Fourier series in Chebyshev polynomials $T_n(x)$ for uniform approximation of functions. Notice that the expansion of a function $f(x)$ in $[-1, 1]$ as a Fourier series in polynomials $T_n(x)$ reduces to the expansion of $f(\cos x)$ in $[-\pi, \pi]$ as a Fourier series in cosines.

For instance,

$$e^{a\cos\varphi} = I_0(a) + 2\sum_{n=1}^{\infty} I_n(a)\cos n\varphi,$$

where $I_n(a)$ is Bessel's function; hence the substitution $x = \cos\varphi$ gives

$$e^{ax} = I_0(a) + 2\sum_{n=1}^{\infty} I_n(a)T_n(x).$$

An expansion may similarly be obtained for $f(x) = x^n$ from the familiar trigonometric formulae:

$$\cos^{2n}\varphi = \frac{1}{2^{2n}}\left(\sum_{k=0}^{n-1} 2C_{2n}^{k}\cos 2(n-k)\varphi + C_{2n}^{n}\right),$$

$$\cos^{2n-1}\varphi = \frac{1}{2^{2n-2}}\sum_{k=0}^{n-1} C_{2n-1}^{k}\cos(2n-2k-1)\varphi.$$

The substitution $x = \cos\varphi$ gives

$$\left.\begin{aligned} x^{2n} &= \frac{1}{2^{2n}}\left(\sum_{k=0}^{n-1} 2C_{2n}^{k}T_{2n-2k}(x) + C_{2n}^{n}\right), \\ x^{2n-1} &= \frac{1}{2^{2n-2}}\sum_{k=0}^{n-1} C_{2n-1}^{k}T_{2n-2k-1}(x). \end{aligned}\right\} \tag{4.140}$$

Similarly, expansions in Chebyshev polynomials $\overline{T}_n(x)$ follow from the expansions in cosines (with $|x| < 1$):

$$|x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{T_{2n}(x)}{4n^2-1}, \tag{4.141}$$

$$\operatorname{sign} x = \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{T_{2n-1}(x)}{2n-1}, \tag{4.142}$$

$$\cos ax = J_0(a) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(a)T_{2n}(x), \tag{4.143}$$

$$\sin ax = 2\sum_{n=1}^{\infty}(-1)^{n+1}J_{2n-1}(a)T_{2n-1}(x), \tag{4.144}$$

$$\arcsin x = \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{T_{2n-1}(x)}{(2n-1)^2}, \tag{4.145}$$

$$e^{ax} = I_0(a)+2\sum_{n=1}^{\infty} I_n(a)T_n(x), \tag{4.146}$$

$$\ln(1-2qx+q^2) = -2\sum_{n=1}^{\infty}\frac{q^n}{n}\overline{T}_n(x) \quad (|q|<1). \tag{4.147}$$

In general, if C_k is the Fourier coefficient of $f(\cos x)$ in the system $\{\cos n\varphi\}$, we have

$$C_k = \frac{2}{\pi}\int_0^{\pi} f(\cos\varphi)\cos n\varphi\, d\varphi = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}. \tag{4.148}$$

Roughly speaking, when such functions are represented by the sum of a like number of terms, the limits of the accuracy are 2^{n-1} times better for an expansion in Chebyshev polynomials than for a Taylor series.

For example, when $|x| \leqq 1$ follows from (4.146) that, in the case of $f(x) = e^{nx}$ with large n,

$$C_n = 2I_n(a) \approx 2\left[\frac{a^n}{n!\,2^n}+\frac{a^{n+2}}{(n+1)!\,2^{n+2}}+\dots\right] \approx \frac{a^n}{n!\,2^{n-1}},$$

while at the same time

$$\frac{f^{(n)}(0)}{n!} = \frac{a^n}{n!}.$$

If we substitute the Maclaurin series for $f(x)$

$$f(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\dots+\frac{f^{(n)}(0)}{n!}x^n+\dots,$$

under the integral sign in (4.148), in view of the orthogonality of the polynomials $T_k(x)$ and the expressions (4.138), (4.140), we get an interesting connection between the coefficient C_n in the expansion of $f(x)$ in Chebyshev polynomials and the corresponding coefficient of Maclaurin's formula for $f(x)$, viz

$$C_n = \frac{1}{2}\left(\frac{f^{(n)}(0)}{n!}+\frac{f^{(n+2)}(0)}{(n+2)!\,1!}+\frac{f^{(n+4)}(0)}{(n+4)!\,2!}+\dots\right). \tag{4.149}$$

It follows from (4.149) that, if the principal part of $f^{(n)}(0)/n! + \dots)$ reduces to $f^{(n)}(0)/n!$ for large n, we have

$$C_n \approx \frac{1}{2}\frac{f^{(n)}(0)}{n!}. \tag{4.150}$$

It follows from (4.150) that, given $|x| < \frac{1}{2}$, the expansion of a function as a Maclaurin series is in general better, whereas the expansion in Chebyshev polynomials is better for $\frac{1}{2} < |x| < 1$, since

$$\max |C_n T_n(x)| = C_n \frac{1}{2^{n-1}} < \max_{\frac{1}{2} < |x| \leqq 1} \left| \frac{f^n(0)}{n!} x^n \right|.$$

The convergence of Fourier series in Chebyshev polynomials. In view of the fact that

$$|T_n(x)| \leqq 1,$$

it follows from Theorem 19 § 3 that:

THEOREM 31. *Given any function $f(x)$, integrable in $(-1, 1)$ with respect to the weight $1/\sqrt{1-x^2}$, its Fourier series in the system $\{T_n(x)\}$ is convergent to the value $f(x)$ at every point x for which the integral*

$$\int_{-1}^{1} \frac{f(x)-f(t)}{x-t} \frac{dt}{\sqrt{1-t^2}}$$

exists. The Lebesgue function of the system of Chebyshev polynomials satisfies the inequality

$$L_n(x) \leqq 2 + \ln n.$$

We therefore have (see Theorem 23 § 3):

THEOREM 32. *Every function $f(x)$ for which*

$$\lim_{n \to \infty} E_n(f) \ln n = 0,$$

can be expanded as a uniformly convergent series in Chebyshev polynomials.

The extremal properties of Chebyshev polynomials.

THEOREM 33 (Chebyshev). *Of all the polynomials with highest coefficient equal to unity, the polynomial $\overline{T}_n(x)$ has the least deviation from zero, i.e.*

$$\max_{-1 \leqq x \leqq 1} |\overline{T}_n(x)| < \max_{-1 \leqq x \leqq 1} |Q_n(x)|$$

for all polynomials $Q_n(x)$ of degree n, having highest coefficient equal to unity.

COROLLARY. *Since*

$$\max_{-1 \leqq x \leqq 1} |\overline{T}_n(x)| = \frac{1}{2^{n-1}},$$

we have for any polynomial $Q_n(x)$ with highest coefficient equal to unity:

$$\max_{-1 \leqq x \leqq 1} |Q_n(x)| > \frac{1}{2^{n-1}}.$$

The following interpolation property of the zeros of a Chebyshev polynomial is a consequence of the theorem. Suppose we form, with respect to the points $x_1, x_2, \ldots, x_n$ an interpolation polynomial $Q_{n-1}(x)$ of degree $n-1$ for a function $f(x)$, n times differentiable in $[-1, 1]$. The remainder term of the accurate interpolation is now given by

$$R_n(x) = f(x) - Q_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x-x_1)(x-x_2)\ldots(x-x_n),$$

where

$$x_i < x_{i+1} \quad \text{and} \quad \xi \in (x_1, x_n).$$

If the interpolation base-points $x_1, x_2, \ldots, x_n$ are zeros of the Chebyshev polynomial $T_n(x)$, then

$$\max_{x \in [-1, 1]} |(x-x_1)(x-x_2)\ldots(x-x_n)|$$

has a minimum value.

Thus, if the coefficient of $f^{(n)}(\xi)$ in (4.148) changes only a little in relation to the variation of x in $(-1, 1)$, the interpolation base-points which are roots of the polynomial $T_n(x)$, yield the least value of the remainder term.

THEOREM 34 (Chebyshev). *Of all the polynomials $Q_n(x)$, subject to the condition $Q_n(\xi) = M$, where $|\xi| < 1$, the polynomial $MT_n(x)/T_n(\xi)$ deviates the least from zero in the segment* $[-1, 1]$.

THEOREM 35 (A. A. Markov and V. A. Markov). *If the polynomial $Q_n(x)$ satisfies, in* $[-1, 1]$, *the inequality*

$$|Q_n(x)| \leqq M,$$

the derivative of order k of this polyomial satisfies in $[-1, 1]$ *the inequality*

$$|Q_n^{(k)}(x)| \leqq M \frac{n^2(n^2-1)\dots[n^2-(k-1)^2]}{1.3.5\dots(2k-1)}, \qquad (4.151)$$

the sign of equality being obtained in (4.151) *only for the polynomial* $T_n(x)$ *at the points* $x = \pm 1$.

8. Chebyshev polynomials of the second kind

The polynomials of the second kind with respect to the weight $1/\sqrt{1-x^2}$ (see § 3, sec. 4), i.e. the numerators of the convergents of the continued fraction (4.137), form an orthogonal system in $[-1, 1]$ with respect to the weight $\varrho(x) = \sqrt{1-x^2}$ and are therefore the Jacobi polynomials corresponding to $\lambda = \mu = \frac{1}{2}$. They are known as *Chebyshev polynomials of the second kind.* We shall denote them by $U_n(x)$.

The Chebyshev polynomials $U_n(x)$ and $T_{n+1}(x)$ are connected by the relationship

$$U_n(x) = C_n \frac{d}{dx} T_{n+1}(x). \qquad (4.152)$$

An explicit expression for the polynomials. We have from (4.152):

$$U_n(x) = C_n \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}. \qquad (4.153)$$

Formula (4.153) defines $U_n(x)$ only in the interval $(-1, 1)$, but this definition may be extended to all x by means of the familiar trigonometric identity

$$\sin(n+1)\varphi = \sin^{n+1}\varphi - C_{n+1}^2 \sin^{n-1}\varphi\cos^2\varphi + \\ + C_{n+1}^4 \sin^{n-3}\varphi\cos^4\varphi - \dots .$$

In $\bar{U}_n(x)$, let the coefficient of x^n be unity, while $\|\hat{U}_n\| = 1$.

Now,

$$\bar{U}_n(x) = \frac{1}{2^n}\,\frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}; \tag{4.154}$$

$$\hat{U}_n(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}. \tag{4.155}$$

Recurrence formula. As the numerators of continued fraction (4.137), the polynomials $\bar{U}_n(x)$ satisfy the same recurrence relation as $T_{n+1}(x)$:

$$\bar{U}_{n+2}(x) = x\bar{U}_{n+1}(x) - \frac{1}{4}\,\bar{U}_n(x),$$

where $\bar{U}_0(x) = 1$, $\bar{U}_1(x) = x$; hence

$$\bar{U}_2(x) = x^2 - \frac{1}{4},$$

$$\bar{U}_3(x) = x^3 - \frac{1}{2}x,$$

$$\bar{U}_4(x) = x^4 - \frac{3}{4}x_2 + \frac{1}{16} \quad \text{and so on.}$$

A continued fraction for polynomials $\bar{U}_n(x)$. These polynomials form an orthogonal system with respect to the weight $\varrho(x) = \sqrt{1-x^2}$ and are in turn the denominators of the nth convergent of the continued fraction

$$\frac{\frac{\pi}{2}}{x} - \frac{\frac{1}{4}}{x} - \frac{\frac{1}{4}}{x} - \ldots, \tag{4.156}$$

where

$$\lambda_0 = \frac{\pi}{2} = \int_{-1}^{1} \sqrt{1-x^2}\,dx, \quad \text{while} \quad \lambda_i = \frac{1}{4} \qquad (i = 1, 2, \ldots).$$

The numerators of the nth convergents of the continued fraction (4.156) are $\bar{U}_{n-1}(x)$. For all x lying outside the segment $[-1, +1]$, the continued fraction is convergent to the function

$$f(x) = (x - \sqrt{x^2-1})\pi.$$

Fourier series in Chebyshev polynomials $U_n(x)$. The following inequalities hold:

$$\left.\begin{aligned} |\hat{U}_n(x)| &< \sqrt{\frac{2}{\pi}}\,(n+1) \qquad &(-1 \leqq x \leqq 1),\\ |\hat{U}_n(x)| &< \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{1-x^2}} \qquad &(-1 < x < 1), \end{aligned}\right\} \tag{4.157}$$

The next theorems are consequences of (4.157) and the general theorems (see § 3, sec. 5).

THEOREM 36. *Every function, having a continuous derivative of the third order, can be expanded as a uniformly convergent series in polynomials* $\hat{U}_n(x)$.

THEOREM 37. *Every function* $f(x)$ *of* $L^2_{\sqrt{1-x^2}}(-1,1)$ *can be expanded as a Fourier series in the orthogonal polynomials* $\hat{U}_n(x)$ *at every point* x, *and for every* $f(x)$ *the following integral exists*:

$$\int_{-1}^{1}\left(\frac{f(x)-f(t)}{x-t}\right)^2\sqrt{1-t^2}\,dt. \tag{4.158}$$

THEOREM 38. *If a function* $f(x)$, *defined on* $[-1,\ 1]$, *satisfies the Dini–Lipschitz condition*

$$\lim_{\delta\to 0}\omega(\delta)\ln\delta = 0, \tag{4.159}$$

it can be expanded in the interval $(-1,\ 1)$ *as a Fourier series in polynomials* $U_n(x)$, *the convergence being uniform in any interval* $(-1+h,\ 1-h)$.

Given suitable convergence conditions, (4.152) enables us to differentiate the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x) \tag{4.160}$$

term by term, to obtain the expansion

$$f'(x) = \sum_{n=1}^{\infty} n a_n U_{n-1}(x),$$

Expansions (4.139–4.142) given above in polynomials T_n lead to the following expansions in polynomials U_n:

$$e^{ax} = \frac{2}{a}\sum_{n=1}^{\infty} nI_n(a)U_{n-1}(x),$$

$$\operatorname{sign} x = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{2nU_{2n-1}(x)}{4n^2-1},$$

$$\sin ax = \frac{2}{a}\sum_{n=1}^{\infty} (-1)^n\, 2nJ_{2n}(a)U_{2n-1}(x),$$

$$\cos ax = \frac{2}{a}\sum_{n=1}^{\infty} (-1)^{n+1}\,(2n-1)J_{2n-1}(a)U_{2(n-1)}(x)$$

and so on.

An extremal property of polynomials $U_n(x)$.

THEOREM 39 (Chebyshev). *Of all the polynomials $Q_n(x)$ of degree n with highest coefficient equal to unity, the least value of the integral*

$$\int_{-1}^{1} |Q_n(x)|\,dx$$

is given by $Q_n(x) = \overline{U}_n(x)$.

9. Laguerre polynomials

The polynomials orthogonal in the interval $(0, +\infty)$ with respect to the weight function $\varrho(x) = x^{\alpha}e^{-x}$ $(\alpha > -1)$ are usually called *Laguerre* or *Chebyshev–Laguerre polynomials.*

These polynomials were first encountered in the case $\alpha = 0$ in the analytical mechanics of Lagrange, then in Abel's posthumous papers. Chebyshev discussed the polynomials in 1859, and for them obtained a recurrence formula and an expansion as a continued fraction; it was only in 1878 that they were considered by Laguerre. The case of any $\alpha > -1$ was first discussed by Sokhotskii.

Some authors only speak of Laguerre polynomials in the case $\alpha = 0$, and refer to *generalized Laguerre polynomials* when $\alpha \neq 0$. We shall denote the latter by $L_n^{\alpha}(x)$.

Rodrigues' formula is

$$L_n^{\alpha}(x) = A_n x^{-\alpha} e^{x} \frac{d^n}{dx^n}(x^{\alpha+n} e^{-x}) \tag{4.161}$$

The polynomial $\bar{L}_n^{\alpha}(x)$ with highest coefficient equal to unity is obtained from (4.161) with $A_n = (-1)^n$, while the normalized $\hat{L}_n^{\alpha}(x)$ is obtained with

$$A_n = \frac{(-1)^n}{\sqrt{n!\,\Gamma(\alpha+n+1)}}.$$

A recurrence formula:

$$\bar{L}_{n+2}^{\alpha}(x) = (x-\alpha-2n-3)\bar{L}_{n+1}^{\alpha}(x)-(n+1)(n+\alpha+1)\bar{L}_n^{\alpha}(x). \tag{4.162}$$

In the case $\alpha = 0$ this reduces to

$$\bar{L}_{n+2}(x) = (x-2n-3)\bar{L}_{n+1}(x)-(n+1)^2\bar{L}_n(x),$$

from which we find, using $\bar{L}_0(x) = 1$, $\bar{L}_1(x) = x-1$,

$$\bar{L}_2(x) = x^2-4x+2,$$

$$\bar{L}_3(x) = x^3-9x^2+18x-6, \quad \text{and so on.}$$

A continued fraction for polynomials $\bar{L}_n^{\alpha}(x)$. In the case $\alpha = 0$, the continued fraction (4.162), of which the denominator of the nth convergent is $\bar{L}_n(x)$, takes the form

$$\frac{1}{x-1-}\,\frac{1^2}{x-3-}\,\frac{2^2}{x-5-}\,\frac{3^2}{x-7-}\cdots. \tag{4.163}$$

The continued fraction (4.163) is convergent for all x not lying in $(0, +\infty)$ to the function

$$f(x) = \int_0^{\infty} \frac{e^{-t}}{x-t}\,dt.$$

The continued fraction for any α,

$$\frac{\Gamma(\alpha+I)}{x-(\alpha+1)-}\,\frac{\alpha+1}{x-(\alpha+3)-}\,\frac{2(\alpha+2)}{x-(\alpha+5)-}\,\frac{3(\alpha+3)}{x-(\alpha+7)}- \tag{4.164}$$

is convergent for any x not lying in $(0, +\infty)$, to the function

$$f(x) = \int_0^{\infty} \frac{e^{-t}t^{\alpha}}{x-t}\,dt.$$

A differential equation for $L_n^\alpha(x)$:

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{4.165}$$

Sokhotskii obtained a formula for the power x^n in terms of Laguerre polynomials:

$$x^n = \Gamma(n+\alpha+1) \sum_{k=0}^{n} \frac{L_{n-k}^\alpha(x)}{\Gamma(n-k+\alpha+1)}.$$

The generating function is

$$(1-t)^{-\alpha-1} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n L_n^\alpha(x)}{n!}. \tag{4.166}$$

Formula (4.166) was first obtained by Sokhotskii. The generating function can also be written in the form

$$\psi_\alpha(t, x) = e^{-t} \frac{I_n(2i\sqrt{tx})}{(i\sqrt{tx})^\alpha} = \sum_{n=0}^{\infty} \frac{L_n^\alpha(x) t^n}{n!\,\Gamma(\alpha+n+1)}, \tag{4.167}$$

where $I_\alpha(z)$ is Bessel's function. Formula (4.167) was obtained by N.Ya. Sonin. Sonin also obtained the relationships

$$\left.\begin{aligned} \frac{1}{(n+1)} \frac{d\overline{L}_{n+1}^\alpha(x)}{dx} &= \overline{L}_n^\alpha(x) - \frac{d\overline{L}_n^\alpha(x)}{dx}, \\ \frac{n(n+\alpha)\overline{L}_{n-1}^{(\alpha)}(x)}{x^{n-1}} &= -\frac{d}{d\frac{1}{x}} \left\{ \frac{\overline{L}_n^\alpha(x)}{x^n} \right\}. \end{aligned}\right\} \tag{4.168}$$

Closure of the system $\{L_n^\alpha(x)\}$. The problem of moments for the sequence

$$\mu_n = \int_0^\infty e^{-x} x^{\alpha+n}\, dx$$

is determinate, from which it follows, by Riesz's Theorem, (see § 3, sec. 7) that the system $\{L_n'(x)\}$ is closed in the space $L^2_{e^{-x}x^\alpha}(0, +\infty)$.

The Laguerre polynomials can be obtained from the Jacobi polynomials $j_n^{(\lambda, \mu)}$ by means of a passage to the limit:

$$L_n^\lambda(x) = \lim_{\mu \to \infty} j_n^{(\lambda, \mu)}\left(1 - \frac{2x}{\mu}\right) \tag{4.169}$$

(K. A. Posset's formula).

10. Hermite polynomials

The polynomials, orthogonal on $(-\infty, +\infty)$ with respect to the weight $\varrho(x) = e^{-\frac{1}{2}x^2}$ are known as *Hermite polynomials.* This name is sometimes given to the polynomials orthogonal with respect to the weight e^{-x^2}. Such polynomials are first encountered in Laplace's works, then in those of Chebyshev (1859). Chebyshev obtained a Rodrigues' formula and an expansion as a continued fraction. They were considered by Hermite five years after Chebyshev. Rodrigues' formula is

$$H_n(x) = A_n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right), \tag{4.170}$$

when $A_n = (-1)^n$ we get $\bar{H}_n(x)$ — the polynomial with highest coefficient equal to unity; when $A_n = (-1)^n/\sqrt{n!}\,\sqrt{2\pi}$ we get the normalized $\hat{H}_n(x)$.

The recurrence formula is

$$\bar{H}_{n+1}(x) = x\bar{H}_n(x) - n\bar{H}_{n-1}(x); \tag{4.171}$$

since $\bar{H}_0(x) = 1, \bar{H}_1(x) = x$, we get

$$\begin{aligned}
\bar{H}_2(x) &= x^2 - 1,\\
\bar{H}_3(x) &= x^3 - 3x,\\
\bar{H}_4(x) &= x^4 - 6x^2 + 3,\\
\bar{H}_5(x) &= x^5 - 10x^3 + 15x, \text{ and so on.}
\end{aligned}$$

The generating function is

$$e^{xt-\frac{t^2}{2}} = \sum_{n=0}^{\infty} \frac{\bar{H}_n(x)t^n}{n!}. \tag{4.172}$$

The continued fraction (4.62), of which the denominator of the nth convergent is the polynomial $H_n(x)$, is

$$\frac{\sqrt{2\pi}}{x\ -}\ \frac{1}{x\ -}\ \frac{2}{x\ -}\ \frac{3}{x\ -\ \dots}. \tag{4.173}$$

Fraction (4.173) is convergent for all non-real x to the function

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{t^2}{2}}}{x-t}\,dt.$$

A differential equation for the Hermite polynomials:

$$y''-xy'+ny = 0. \tag{4.174}$$

The polynomials $H_n(x)$ can also be obtained from the Jacobi polynomials $j_n^{(\lambda,\lambda)}(x)$ as $\lambda \to \infty$:

$$\frac{H_n(x)}{n!} = \lim_{\lambda \to \infty}\left[\lambda^{-\frac{n}{2}} j_n^{(\lambda,\lambda)}\left(\frac{x}{\sqrt{2\lambda}}\right)\right]. \tag{4.175}$$

Closure of the system of Hermite polynomials. The problem of moments for the sequence

$$\mu_n = \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}\,dx$$

is determinate. Hence it follows by Riesz's theorem (see § 3, sec. 7) that the system $\{H_n(x)\}$ is closed in the space $L^2_{e^{-\frac{1}{2}x^2}}(-\infty, +\infty)$.

11. Chebyshev polynomials, orthogonal on a finite system of points

In a paper of 1855, "Continuous fractions" (O nepreryvnykh drobyakh), and in certain other works, Chebyshev considered the expansion of the sum

$$\sum_{i=1}^{N} \frac{\theta^2(x_i)}{z-x_i}, \qquad \theta^2(x_i) = m_i$$

as a continued fraction and investigated the properties of the denominators of its convergents.

If $\sigma(x)$ is a step function with growth points $x_1, x_2, \ldots, x_N$, the jumps at which are equal to $m_1, m_2, \ldots, m_N$ respectively, then

$$\sum_{i=1}^{N} \frac{\theta^2(x_i)}{z-x_i} = \int_{-\infty}^{\infty} \frac{d\sigma(x)}{2-x}. \tag{4.176}$$

The denominators of the convergents of the continued fraction (of type (4.62)), corresponding to integral (4.176), are now polynomials orthogonal with respect to the weight function $\sigma(x)$.

The orthogonality of $P_s(x)$ and $P_k(x)$ with respect to $\sigma(x)$ implies in this case that

$$\sum_{i=1}^{N} \theta^2(x_i)P_s(x_i)P_k(x_i) = 0 \qquad s \neq k.$$

The continued fraction (4.62) is now finite, the system of orthogonal polynomials is also finite and contains precisely N polynomials $P_0(x)$, $P_1(x)$, $\ldots$, $P_{N-1}(x)$. Chebyshev applied the results of his investigations to interpolation by the method of least squares which consists of the following. The values of the function $f(x)$ are specified at the points x_1, x_2, $\ldots$, x_N. Among all the polynomials of a given degree $n < N$, we have to find the polynomial $Q_n(x)$ such that the sum

$$\sum_{i=1}^{N} [f(x_i) - Q_n(x_i)]^2 \tag{4.177}$$

is a minimum (see § 1, sec. 3).

Let $\theta^2(x_i) = 1$ in (4.176); then

$$\sum_{i=1}^{N} [f(x_i) - Q_n(x_i)]^2 = \int_{-\infty}^{\infty} (f(x) - Q_n(x))^2 \, d\sigma(x).$$

And, as follows from § 2, sec. 4, (4.177) takes its least value when $Q_n(x)$ is the nth segment of the Fourier series of $f(x)$ in a system of polynomials orthogonal with respect to the weight function $\sigma(x)$.

Using a system of equidistant base points on the segment [0, 1]:

$$x_i = \frac{i}{N} \qquad (i = 1, 2, \ldots, N).$$

Chebyshev formed an orthogonal system of polynomials $\{\theta(x_i)^2 = 1\}$. These are also known as *Chebyshev polynomials*. We shall denote them by $P_{k,N}(x)$ (the polynomial of degree k in the system $\{x_i = i/N\}$):

$$P_{k,N}(x) = 1 + a_1 x + a_2 x(x-1) + a_3 x(x-1)(x-2) + \ldots$$
$$\ldots + a_k x(x-1)\ldots(x-k+1),$$

where

$$a_s = \frac{(-1)^s C_k^s C_{k+1}^s}{n(n-1)\dots(n-s+1)}.$$

In particular,

$$P_{0,N}(x) = 1,$$

$$P_{1,N}(x) = 1-2\frac{x}{N},$$

$$P_{2,N}(x) = 1-6\frac{x}{N}+6\frac{x(x-1)}{N(N-1)},$$

$$P_{3,N}(x) = 1-12\frac{x}{N}+30\frac{x(x-1)}{N(N-1)}-20\frac{x(x-1)(x-2)}{N(N-1)(N-2)},$$

$$P_{4,N}(x) = 1-20\frac{x}{N}+90\frac{x(x-1)}{N(N-1)} - $$

$$-140\frac{x(x-1)(x-2)}{N(N-1)(N-2)}+70\frac{x(x-1)(x-2)(x-3)}{N(N-1)(N-2)(N-3)},$$

$$P_{5,N}(x) = 1-30\frac{x}{N}+210\frac{x(x-1)}{N(N-1)} - $$

$$-560\frac{x(x-1)(x-2)}{N(N-1)(N-2)}+630\frac{x(x-1)(x-2)(x-3)}{N(N-1)(N-2)(N-3)} - $$

$$-252\frac{x(x-1)(x-2)(x-3)(x-4)}{N(N-1)(N-2)(N-3)(N-4)},$$

and so on. The interpolation polynomial of degree m with respect to the system base points $\{x_i = i/N\}$ for $f(x)$, defined by the method of least squares, is the segment of the Fourier series of $f(x)$ of order m with respect to the system $\{P_{k,N}(x)\}$:

$$Q_m(x) = \sum_{k=0}^{m} C_k P_{k,N}(x),$$

where

$$C_k = \int_0^1 f(x)P_{k,N}(x)\,d\sigma(x) = \sum_{i=1}^{N} f(x_i)P_{k,N}(x_i).$$

The expression for C_k does not depend on the degree of interpolation m. On increasing the degree of the interpolation polynomial new terms are simply added:

$$Q_{m+1}(x) = Q_m(x) + C_{m+1}P_{m+1,\,N}(x)\,.$$

Chebyshev showed that the case when $\sigma(x)$ is continuous can be obtained by means of a passage to the limit from the case when $\sigma(x)$ has a finite number of growth points.

In particular, the Legendre polynomials can be obtained from the polynomials $P_{k,\,N}(x)$ as $N \to \infty$. If $L_k(x)$ is a Legendre polynomial, normalized by the condition that $L_k(1) = 1$, we have

$$L_k(2x-1) = \frac{(2k)!}{(k!)^2} \lim_{N\to\infty} \left\{ \frac{P_{k,\,N}(Nx)}{N^k} \right\} \qquad (k = 1,\ 2,\ \ldots). \tag{4.178}$$

CHAPTER V

CONTINUED FRACTIONS

Introduction

1. Notation for continued fractions. Basic definitions

An expression of the form

$$b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\ddots+\cfrac{a_n}{b_n+\ddots}}}$$

is called a *continued fraction.*

In view of the unwieldiness of the above method of writing, several authors have proposed different methods, for instance:

$$b_0+\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\dots+\frac{a_n|}{|b_n}+\dots, \qquad \text{(Pringsheim)}$$

$$b_0+\frac{a_1}{b_1}\dot{+}\frac{a_2}{b_2}\dot{+}\dots\dot{+}\frac{a_n}{b_n}\dot{+}\dots, \qquad \text{(Müller)}$$

$$b_0+\frac{a_1}{b_1+}\frac{a_2}{b_2+}\dots\frac{a_n}{+b_n+}\dots. \qquad \text{(Rogers)} \tag{5.1}$$

We shall use the last notation. Pringsheim also proposed writing a continued fraction in the form $[b_0;\ a_\nu/b_\nu]_1^\infty$. If a_1, and b_1 in fraction (5.1) follow a different law to the remaining a_n and b_n, Pringsheim used the notation $[b_0;\ a_1/b_1,\ a_\nu/b_\nu]_2^\infty$.

The fraction a_n/b_n is called the *n-th partial quotient* of the continued fraction (5.1); a_n and b_n are the *terms* of the *n-th* partial quotient; $a_1, a_2, a_3, \dots$ are called the *partial numerators* of the continued fraction; $b_1, b_2, b_3, \ldots$ are its *partial denominators*; b_0 is called the *zero partial quotient*.

All the terms of the partial quotients will be assumed finite; all the partial denominators are usually assumed to be non-zero.

A continued fraction having a finite set of partial quotients is described as *terminating*.

A continued fraction having an infinite set of partial quotients is described as *non-terminating*.

2. A brief historical note

Algorithms similar to continued fractions were employed even by the mathematicians of antiquity (Euclid's algorithm, the Archimedean approximation to $\sqrt{3}$). Of the mediaeval mathematicians, Omar Khayyam (c. 1040–1123) came close to continued fractions when trying to generalize Euclid's algorithm to the case of incommensurable magnitudes, but continued fractions as such first appeared in the "Algebra" of the Italian mathematician R.Bombelli, published in 1572. A number of outstanding mathematicians of the seventeenth century, including J. Wallis and Chr. Huygens, occupied themselves with continued fractions, but the founder of the theory of continued fractions as an independent branch of mathematics was Euler. Almost all the great mathematicians of the eighteenth century and the first half of the nineteenth century made some contribution to the development of the theory. Interest has recently returned afresh to continued fractions, owing to their great theoretical and practical value. In particular, continued fractions are employed in various approximate computations. For instance, with their aid we can compute approximately the values of many functions, the power series expansion of which are slowly convergent or even divergent.

§ 1. Continued fractions and their fundamental properties

1. The evaluation of convergents. Convergents

The terminating continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} \equiv \frac{P_n}{Q_n}$$

is called the *n*th *convergent* of the continued fraction (5.1). We put here

$$\frac{P_0}{Q_0} = \frac{b_0}{1}, \quad \frac{P_{-1}}{Q_{-1}} = \frac{1}{0}.$$

Basic recurrence relations:

$$\left.\begin{aligned} P_n &= b_nP_{n-1}+a_nP_{n-2}, \\ Q_n &= b_nQ_{n-1}+a_nQ_{n-2} \end{aligned} \quad (n = 1, 2, 3, \ldots). \right\} \tag{5.2}$$

Equations (5.2) enable the convergents to be evaluated successively. It is useful here to employ the following scheme:

$$\begin{array}{l} b_0+\dfrac{a_1}{b_1}+\dfrac{a_2}{b_2}+\ldots+\dfrac{a_n}{b_n}+\ldots, \\ \dfrac{1}{0}\;\;\dfrac{b_0}{1}\;\;\dfrac{P_1}{Q_1}\;\;\dfrac{P_2}{Q_2}\;\;\cdots\;\;\dfrac{P_n}{Q_n}\cdots. \end{array}$$

EXAMPLE 1.

$$\sqrt{2} = 1+(\sqrt{2}-1) = 1+\frac{1}{1+\sqrt{2}} =$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$$

$$\begin{array}{ccccccc} \dfrac{1}{0} & \dfrac{1}{1} & \dfrac{3}{2} & \dfrac{7}{5} & \dfrac{17}{12} & \dfrac{41}{29} & \dfrac{99}{70} \\ & 1 & 1{\cdot}5 & 1{\cdot}4 & 1{\cdot}417 & 1{\cdot}4138 & 1{\cdot}41429. \end{array}$$

The difference between neighbouring convergents is

$$\frac{P_n}{Q_n}-\frac{P_{n-1}}{Q_{n-1}} = (-1)^{n+1}\frac{a_1a_2\ldots a_n}{Q_{n-1}Q_n} \qquad (n = 1, 2, 3, \ldots). \tag{5.3}$$

The difference between convergents whose indices differ by 2 is:

$$\frac{P_{n+1}}{Q_{n+1}}-\frac{P_{n-1}}{Q_{n-1}} = (-1)^{n+1}\frac{a_1a_2\ldots a_nb_{n+1}}{Q_{n-1}Q_{n+1}} \qquad (n = 1, 2, 3, \ldots). \tag{5.4}$$

Continued fractions whose partial quotients have positive terms. It follows from equations (5.4) that, if all the terms of the partial quotients are positive, the convergents of even order form a monotonically increasing sequence, bounded from above by the number b_0+a_1/b_1. Such a sequence has a limit. Consequently, $\lim_{n\to\infty} P_{2n}/Q_{2n}$ exists in this case. Similarly, if all the terms of the partial quotients are positive, the convergents of odd order form a monotonically decreasing sequence, bounded from below by the number b_0. Such a sequence also has a limit. Thus $\lim_{n\to\infty} P_{2n-1}/Q_{2n-1}$ also exists. Thus the value (if it exists) of a continued fraction, of which all the terms of the partial quotients are positive, is always greater than any convergent of even order and less than any convergent of odd order. The number $\lim_{n\to\infty} P_n/Q_n$ is taken to be the value of a continued fraction (5.1).

2. Transformations of continued fractions

The fundamental identity transformation is

$$b_0+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\dots+\frac{a_n}{b_n}=b_0+$$

$$+\frac{p_1a_1}{p_1b_1}+\frac{p_1p_2a_2}{p_2b_2}+\dots+\frac{p_{n-1}p_na_n}{p_nb_n}+\dots, \qquad (5.5)$$

where $p_1, p_2, \dots$ are any numbers, finite and non-zero.

Ordinary continued fractions. By using transformation (5.5), we can always reduce a continued fraction (5.1) to the form

$$\alpha_0+\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\dots+\frac{1}{\alpha_n}+\dots, \qquad (5.6)$$

where

$$\alpha_0=b_0; \quad \alpha_{2k-1}=\frac{a_2a_4\dots a_{2k-2}b_{2k-1}}{a_1a_3\dots a_{2k-1}}, \quad \alpha_{2k}=\frac{a_1a_3\dots a_{2k-1}b_{2k}}{a_2a_4\dots a_{2k}}.$$

We call (5.6) an *ordinary continued fraction*, while the numbers α_1, $\alpha_2, \dots$ are the *partial denominators* of the ordinary continued fraction.

An ordinary continued fraction with positive integral partial denominators is said to be *regular*. Only regular continued fractions are usually considered in the theory of numbers.

Continued fractions, of which all the partial denominators are equal to 1. Such fractions are obtained from the fraction (5.1) by a transformation (5.5) in which we put $p_n = 1/b_n (n = 1, 2, \dots)$. When $b_0 = 0$, such a fraction has the form

$$\frac{c_1}{1}+\frac{c_2}{1}+\dots+\frac{c_n}{1}+\dots, \qquad (5.7)$$

where

$$c_1 = \frac{a_1}{b_1}, \quad c_n = \frac{a_n}{b_{n-1}b_n} \quad (n = 2, 3, \ldots).$$

The terms of the partial quotients of fractions (5.6) and (5.7) are connected by the relationships

$$c_1 = \frac{1}{\alpha_1}, \quad c_n = \frac{1}{\alpha_{n-1}\alpha_n} \qquad (n = 2, 3, \ldots).$$

Daniel Bernoulli's continued fraction. The continued fraction, the convergents of which are equal to $K_0, K_1, K_2, \ldots$, has the form

$$K_0 + \frac{K_1-K_0}{1\;+}\,\frac{K_1-K_2}{K_2-K_0+}\,\frac{(K_1-K_0)(K_2-K_3)}{K_3-K_1\quad+}\ldots$$
$$\ldots+\frac{(K_{n-2}-K_{n-3})(K_{n-1}-K_n)}{K_n-K_{n-2}\quad+\ldots}. \tag{5.8}$$

EXAMPLE 2. The continued fraction for which $K_n = 1/(n+1)^2$ $(n = 0, 1, \ldots)$, has the form

$$K = 1 - \frac{3}{4.1\,-}\,\frac{1.5}{4.2\,-}\,\frac{3.7}{4.3\,-}\,\frac{5.9}{4.4\,-}\;\ldots\;-\frac{(2n-3)(2n+1)}{4n\quad-\ldots}.$$

$$\frac{1}{1}\quad\frac{1}{4}\quad\frac{3}{27}\quad\frac{15}{240}\ldots$$

3. Contraction and extension of continued fractions

If we take as $K_0, K_1, K_2, \ldots$ some subsequence of convergents of fraction (5.1), we say that the fraction (5.8) has been obtained by means of a *contraction of fraction* (5.1). Whereas, if we take as $K_0, K_1, K_2, \ldots$ a sequence which includes in particular all the convergents of (5.1), we say that the fraction (5.8) has been obtained by means of an *extension of fraction* (5.1).

After putting $K_n = P_{2n}/Q_{2n}$, we have

$$b_0 + \frac{a_1}{b_1\,+}\,\frac{a_2}{b_2\,+}\,\ldots\,\frac{a_n}{+b_n\,+}\ldots = b_0 + \frac{a_1b_2}{b_1b_2+a_2\,-}$$
$$-\frac{a_2a_3b_4}{(b_2b_3+a_3)b_4+b_2a_4\,-}\,\frac{a_4a_5b_2b_6}{(b_4b_5+a_5)b_6+b_4a_6\,-}\ldots$$
$$\ldots-\frac{a_{2n-2}a_{2n-1}b_{2n-4}b_{2n}}{(b_{2n-2}b_{2n-1}+a_{2n-1})b_{2n}+b_{2n-2}a_{2n}\,-\ldots}. \tag{5.9}$$

EXAMPLE 3.

$$\sqrt{2} = 1+\frac{1}{2+}\frac{1}{2+}\dots = 1+\frac{2}{5-}\frac{1}{6-}\frac{1}{6-}\dots\frac{1}{6-}\dots.$$

$$\frac{1}{7}\quad\frac{7}{5}\quad\frac{41}{29}\quad\frac{239}{169}$$

On putting $K_n = P_{2n+1}/Q_{2n+1}$, we have

$$b_0+\frac{a_1}{b_1+}\frac{a_2}{b_2+}\dots\frac{a_n}{b_n+}\dots \overset{*}{=} \frac{b_0b_1+a_1}{a_1} - \frac{a_1a_2b_3}{(b_1b_2+a_2)b_3+b_1a_3-}\frac{a_3a_4b_1b_5}{(b_3b_4+a_4)b_5+b_3a_5-}\dots$$

$$\dots-\frac{a_{2n-1}a_{2n}b_{2n-3}b_{2n+1}}{(b_{2n-1}b_{2n}+a_{2n})b_{2n+1}+b_{2n-1}a_{2n+1}-}\dots.$$

Here, the sign $\overset{*}{=}$ indicates that the convergent of zero order of the continued fraction on the right-hand side of the last equation is the fraction 1/0, and not 0/1, since this continued fraction has for its convergents the convergents of odd order only of continued fraction (5.1).

EXAMPLE 4.

$$1+\sqrt{2} = 2+\frac{1}{2+}\dots\frac{1}{2+}\dots \overset{*}{=} \frac{5}{2-}\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\dots.$$

$$\frac{1}{0}\quad\frac{5}{2}\quad\frac{29}{12}\quad\frac{169}{70}\quad\frac{985}{408}$$

If we replace the sign $\overset{*}{=}$ by the usual equality, we get the expansion

$$\frac{5}{7}(2\sqrt{2}+1) = \frac{5}{2-}\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\dots.$$

$$\frac{0}{1}\quad\frac{5}{2}\quad\frac{30}{11}\quad\frac{175}{64}\quad\frac{1020}{373}$$

The connection between ordinary and singular values of a continued fraction. If $b_0 = 0$, the continued fraction (5.1) can be considered in two ways, depending on whether we take its zero order convergent as equal to 0/1 or 1/0. In the first case the values of the continued fraction are described as *ordinary*, and in the second as *singular*. On writing K and $\tilde{K}$ respectively for the ordinary and singular value of the same continued fraction, we have

$$K = \frac{a_1}{b_1 +}\frac{a_2}{b_2 +}\dots, \qquad \tilde{K} \overset{*}{=} \frac{a_1}{b_1 +}\frac{a_2}{b_2 +}\dots.$$

$$\frac{0}{1}\quad \frac{a_1}{b_1}\quad \frac{a_1 b_2}{b_1 b_1 + a_2} \qquad\qquad \frac{1}{0}\quad \frac{a_1}{b_1}\quad \frac{a_1 b_2 + a_2}{b_1 b_2}$$

The continued fraction for which $\tilde{K}$ is the ordinary value is

$$\tilde{K} = \frac{a_1}{b_1 -}\frac{\frac{b_1 a_2}{a_1}}{b_2 + \frac{a_2}{a_1} +}\frac{a_3}{b_3 +}\frac{a_4}{b_4 +}\dots.$$

$$\frac{0}{1}\quad \frac{a_1}{b_1}\quad \frac{a_1 b_2 + a_2}{b_1 b_2}$$

It follows that the connection between K and $\tilde{K}$ is given by

$$K = \frac{a_1}{b_1 \tilde{K} + b_1 - \alpha_1}.$$

4. The transformation of a continued fraction resulting from a theorem of Stolz

The following is well known in mathematical analysis:

STOLZ'S THEOREM. *Let*

$$\lim_{n \to \infty} P_n = \infty, \quad \lim_{n \to \infty} Q_n = \infty \; Q_{n+1} > Q_n$$

for all n. Then

$$\lim_{n \to \infty} \frac{P_n}{Q_n} = \lim_{n \to \infty} \frac{P_{n+1} - P_n}{Q_{n+1} - Q_n},$$

if the limit on the right-hand side of the last equation exists.

By using this theorem and the basic recurrence relations (5.2), we can transform the fraction (5.1) into a continued fraction, whose nth order convergent is $(P_n - P_{n-1})/(Q_n - Q_{n-1})$, where P_n and

P_{n-1} are respectively the numerators of the nth and $(n-1)$th order convergents of the continued fraction (5.1), while Q_n and Q_{n-1} are the denominators of these convergents. This continued fraction has the form

$$b_0+\frac{a_1}{b_1}+\ldots+\frac{a_n}{b_n}+\ldots = b_0+\frac{a_1}{b_1-1}+\frac{a_2+b_2-1}{b_2-1}+$$

$$\frac{\dfrac{a_3+b_3-1}{a_2+b_2-1}a_2}{+\ b_3-1+\dfrac{a_3+b_3-1}{a_2+b_2-1}}+\ldots+\frac{\dfrac{a_n+b_n-1}{a_{n-1}+b_{n-1}-1}a_{n-1}}{b_n-1+\dfrac{a_n+b_n-1}{a_{n-1}+b_{n-1}-1}}+\ldots.$$

EXAMPLE 5.

$$\sqrt{3} = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}+\ldots =$$

$$\frac{1}{1}\quad\frac{2}{1}\quad\frac{5}{3}\quad\frac{7}{4}\quad\frac{19}{11}$$

$$= 1+\frac{1}{0}+\frac{2}{1}+\frac{1}{1}+\frac{4}{3}+\frac{1}{1}+\frac{4}{3}+\ldots.$$

$$\frac{1}{1}\quad\frac{1}{0}\quad\frac{3}{2}\quad\frac{4}{2}\quad\frac{24}{14}$$

A more general transformation of continued fraction following from Stolz's theorem. We apply Stolz's theorem to the sequence

$$\left\{\frac{\gamma_n P_n}{\gamma_n Q_n}\right\} \qquad (n = 0, 1, 2, \ldots),$$

where P_n/Q_n $(n = 0, 1, 2, \ldots)$ are the convergents of (5.1), and γ_0 $\gamma_1, \gamma_2, \ldots$ are any non-zero numbers. Now, if the inequality $\gamma_n Q_n > \gamma_{n-1}Q_{n-1}$ is satisfied for all n, we have, by Stolz's Theorem:

$$\lim_{n\to\infty}\frac{P_n}{Q_n} = \lim_{n\to\infty}\frac{\gamma_n P_n-\gamma_{n-1}P_{n-1}}{\gamma_n Q_n-\gamma_{n-1}Q_{n-1}},$$

if the limit on the right-hand side exists.

By using this equation and the basic recurrence relations (5.2), we can transform (5.1) into a continued fraction, whose nth order convergent is $(\gamma_n P_n - \gamma_{n-1}P_{n-1})/(\gamma_n Q_n - \gamma_{n-1}Q_{n-1})$. This continued fraction has the form

$$K = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} + \ldots =$$

$$= b_0 + \frac{\gamma_1 a_1}{\gamma_1 b_1 - \gamma_0 +} \; \frac{\frac{1}{\gamma_1}(\gamma_1\gamma_2 a_2 + \gamma_0\gamma_2 b_2 - \gamma_0\gamma_1)}{\frac{\gamma_2}{\gamma_1} b_2 - 1} \quad +$$

$$\frac{\dfrac{\gamma_2\gamma_3 a_3 + \gamma_1\gamma_3 b_3 - \gamma_1\gamma_2}{\gamma_1\gamma_2 a_2 + \gamma_0\gamma_2 b_2 - \gamma_0\gamma_1}\, a_2}{+\dfrac{\gamma_3 b_3 - \gamma_2}{\gamma_2} + \dfrac{\gamma_0}{\gamma_2}\, \dfrac{\gamma_2\gamma_3 a_3 + \gamma_1\gamma_3 b_3 - \gamma_1\gamma_2}{\gamma_1\gamma_2 a_2 + \gamma_0\gamma_2 b_2 - \gamma_0\gamma_1}} + \ldots$$

$$\ldots + \frac{\dfrac{\gamma_{n-1}\gamma_n a_n + \gamma_{n-2}\gamma_n b_n - \gamma_{n-2}\gamma_{n-1}}{\gamma_{n-2}\gamma_{n-1} a_{n-1} + \gamma_{n-3}\gamma_{n-1} b_{n-1} - \gamma_{n-3}\gamma_{n-2}}\, a_{n-1}}{\dfrac{\gamma_n b_n - \gamma_{n-1}}{\gamma_{n-1}} + \dfrac{\gamma_{n-3}}{\gamma_{n-1}}\, \dfrac{\gamma_{n-1}\gamma_n a_n + \gamma_{n-2}\gamma_n b_n - \gamma_{n-2}\gamma_{n-1}}{\gamma_{n-2}\gamma_{n-1} a_{n-1} + \gamma_{n-3}\gamma_{n-1} b_{n-1} - \gamma_{n-3}\gamma_{n-2}}} + \ldots .$$

EXAMPLE 6.

$$(\gamma_n = n+1; \quad n = 0, 1, 2, \ldots)$$

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} + \ldots =$$

$$= 1 + \frac{2}{3} + \frac{5}{2} + \frac{11}{12} + \frac{95}{26} + \frac{319}{44} + \ldots + \frac{(n^3 - 3n + 1)(n^2 + n - 1)}{2(n^2 - 3)} + \ldots$$

$$\frac{1}{1} \quad \frac{5}{3} \quad \frac{15}{11} \quad \frac{235}{165} \quad \frac{7535}{5335}$$

This gives us an expansion of $\sqrt{2}$ as a non-periodic continued fraction. In number theory there is a theorem which asserts that every irrational square root can be expanded as a periodic continued fraction and that, conversely, the value of every convergent periodic continued fraction is some irrational square root. But it is not always pointed out that an irrational square root can also be expanded as a *non-periodic* continued fraction, there being an infinity of such expansions. The expansion of $\sqrt{2}$ obtained by us as a periodic continued fraction is more slowly convergent than the initial expansion of $\sqrt{2}$.

Corollaries of Stolz's theorem and transformations of continued fractions following from them. Let the sequence $\{P_n/Q_n\}$ satisfy the conditions of Stolz's theorem. The sequence $\{P_n^2/Q_n^2\}$ now also satisfies the conditions of the theorem. Therefore

$$\lim_{n\to\infty}\frac{P_n}{Q_n}=\lim_{n\to\infty}\frac{P_n+P_{n-1}}{Q_n+Q_{n-1}}.$$

We can obtain, on the basis of this equation, the following transformation of a continued fraction:

$$b_0+\frac{a_1}{b_1}+\ldots+\frac{a_n}{b_n}+\ldots=b_0+\frac{a_1}{b_1+1}-$$

$$-\frac{b_2-a_2+1}{b_2+1}+\frac{\dfrac{b_3-a_3+1}{b_2-a_2+1}a_2}{b_3+1-\dfrac{b_3-a_3+1}{b_2-a_2+1}}+\ldots+\frac{\dfrac{b_n-a_n+1}{b_{n-1}-a_{n-1}+1}a_{n-1}}{b_n+1-\dfrac{b_n-a_n+1}{b_{n-1}-a_{n-1}+1}+\ldots}.$$

EXAMPLE 7.

$$\sqrt{3}=1+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}+\ldots=$$

$$=1+\frac{1}{2}-\frac{2}{3}+\frac{1}{3}+\frac{4}{4}+\frac{1}{3}+\frac{4}{1}+\ldots$$

The following equation may also readily be obtained from Stolz's theorem:

$$\lim_{n\to\infty}\frac{P_n}{Q_n}=\lim_{n\to\infty}\frac{\gamma_nP_n\pm\gamma_{n-k}P_{n-k}}{\gamma_nQ_n\pm\gamma_{n-k}Q_{n-k}},$$

where $\{\gamma_n\}$ is a sequence, the terms of which are all non-zero. We can obtain on the basis of this equation as many different identity transformations of a continued fraction as desired.

A further transformation of continued fractions. Let us consider the identity

$$b_0+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots=\frac{b_0}{1}+\frac{c_1}{d_1}+\frac{c_2}{d_2}+\ldots \tag{5.10}$$

Let us suppose that the convergents of even order of the fractions appearing in this identity are equal if their indices are the same. Now, on performing a contraction of the continued fractions and using the basic identity transformation to make corresponding terms of partial quotients of the contracted fractions equal, we obtain the following two systems of equations, connecting the terms of the partial quotients of the initial continued fractions:

$$\begin{aligned} -b_0c_1d_2 &= a_1b_2, \\ c_2c_3d_4 &= a_2a_3b_4, \end{aligned} \tag{5.11}$$

and $$c_{2n-2}c_{2n-1}d_{2n} = a_{2n-2}a_{2n-1}b_{2n-4}b_{2n} \quad n = 2, 3, \dots .$$

$$\left.\begin{aligned} &c_1d_2+d_1d_2+c_2 = b_1b_2+a_2, \\ &(d_{2n-2}d_{2n-1}+c_{2n-1})d_{2n}+d_{2n+2}c_{2n} = \\ &= (b_{2n-2}b_{2n-1}+a_{2n-1})b_{2n}+b_{2n-2}a_{2n} \quad (n = 2, 3, \dots). \end{aligned}\right\} \tag{5.12}$$

Systems (5.11) and (5.12), taken together, contain $2n$ equations with $4n$ unknowns c_n and d_n $(n = 1, 2, \dots)$. It is thus impossible, in the general case of equations (5.11) and (5.12), to express c_n and d_n $(n = 1, 2, \dots)$ in terms of $a_1, a_2, \dots, b_1, b_2, \dots$ or, conversely, a_n, b_n $(n = 1, 2, \dots)$ in terms of $c_1, c_2, \dots, d_1, d_2, \dots$. But if $b_0 = 1$, $c_n = -a_n$, $d_{2n} = b_{2n}$, $d_{2n-1} = \lambda_n b_{2n-1}$ $(n = 1, 2, \dots; \lambda_n \neq 1)$, the identity (5.10) takes the form

$$\begin{aligned} &1+\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\dots\frac{a_n}{b_n+}\dots = \\ &= \frac{1}{1-}\,\frac{a_1b_2}{a_1b_2+b_1b_2+2a_2-}\,\frac{a_2}{1-}\,\frac{a_3b_4}{b_2b_3b_4+2a_3b_4+2a_4b_2-}\,\frac{a_4b_2}{1-}\dots \\ &\dots-\frac{a_{2n-1}b_{2n}}{b_{2n-2}b_{2n-1}b_{2n}+2a_{2n-1}b_{2n}+2a_{2n}b_{2n-2}-}\,\frac{a_{2n}b_{2n-2}}{1-}\dots. \end{aligned} \tag{5.13}$$

EXAMPLE 8.

$$\sqrt{2} = 1+\frac{1}{2+}\,\frac{1}{2+}\dots = \frac{1}{1-}\,\frac{2}{8-}\,\frac{1}{1-}\,\frac{1}{8-}\,\frac{1}{1-}\,\frac{1}{8-}\,\frac{1}{1-}\dots.$$

$$\frac{0}{1}\quad\frac{1}{1}\quad\frac{8}{6}\quad\frac{7}{5}\quad\frac{48}{34}\quad\frac{41}{29}\quad\frac{280}{198}\quad\frac{239}{169}$$

If the equations

$$b_0 = 1, \; a_1 = \frac{a_3 - a_2}{b_2} = \frac{a_5 - a_4}{b_4} = \ldots = \frac{a_{2n+1} - a_{2n}}{b_{2n}} = \ldots, \qquad (5.14)$$

are satisfied, then equation (5.10) takes the form

$$1 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} + \ldots =$$

$$= \frac{1}{1} - \frac{a_1}{b_1} + \frac{a_3}{b_2} + \frac{a_2}{b_3} + \ldots + \frac{a_{2n+1}}{b_{2n}} + \frac{a_{2n}}{b_{2n+1}} + \ldots. \qquad (5.15)$$

5. The properties of regular continued fractions

A continued fraction in which all the partial numerators are equal to unity, and all the partial denominators are positive integers, is described as *regular*, or *arithmetical*. It follows from equation (5.3) that all the convergents of a regular continued fraction are non-reducible.

Euclid's algorithm and the expansion of rational numbers as regular continued fractions. Let u and v be two given positive integers, where $u > v$. On dividing u by v, we have

$$\frac{u}{v} = q_0 + \frac{u_1}{v},$$

where q_0 is the quotient and u_1 the remainder. On dividing v by u_1, we have similarly

$$\frac{v}{u_1} = q_1 + \frac{u_2}{u_1}.$$

Proceeding in this manner, we get

$$\frac{u_1}{u_2} = q_2 + \frac{u_3}{u_2}$$

and so on. This process of successive divisions is known as *Euclid's algorithm*. Since $u, u_1, u_2, \ldots$ is a monotonically decreasing sequence of positive integers, the process is finite, i.e. there exists an index n such that $u_{n-1}/u_n = q_n$ (consequently, $u_n \neq 0$, $u_{n+1} = 0$). Hence u_n is the *greatest common divisor* of the numbers u and v. It is often denoted by (u, v).

EXAMPLE 9. Find the greatest common divisor of the numbers 816 and 323.

The computation is usually set out thus:

$$\begin{array}{rrrrr}
 & & & 816 & \mid \underline{323} \\
 & & - & \underline{646} & 2 \\
 & & 323 & \mid \underline{170} & \\
 & - & \underline{170} & 1 & \\
 & \underline{170} & \mid \underline{153} & & \\
\underline{153} & \mid \underline{17} & 1 & & \\
0 & 9 & & &
\end{array}$$

(816, 323) = 17. Thus the expansion of 816/323 as a regular continued fraction has the form

$$\frac{816}{323} = 2+\frac{1}{1}{+}\frac{1}{1}{+}\frac{1}{9}$$

$$\frac{2}{1} \quad \frac{3}{1} \quad \frac{5}{2} \quad \frac{48}{19}.$$

Thus Euclid's algorithm enables us not only to find the greatest common divisor of two positive integers, but also to expand their ratio as a regular continued fraction.

The solution in integers of an indeterminate equation of the first degree with the aid of Euclid's algorithm. The equation $ax+by = c$, where a, b and c are known, while x and y are unknown, is known as an *indeterminate equation of the first degree.* Such an equation has an infinite set of solutions. But it is often required to find only the integers that satisfy the equation, i e. in conventional language, it is required to solve the equation *in integers.* We consider here only equations for which a, b, c are integers. The equation $ax+by = c$ has an integral solution only in the case when c is divisible by (a, b). We can thus always assume that a and b are mutually prime. In this case the general solution in integers of the equation has the form

$$x = (-1)^{n-1}cQ_{n-1}+tQ_n,$$
$$y = (-1)^{n}cP_{n-1}-tP_n,$$

where t is an arbitrary integer, P_{n-1} and Q_{n-1} are the numerator and denominator of the penultimate convergent in the expansion of $a/b = P_n/Q_n$ as a regular continued fraction.

EXAMPLE 10. $43x+37y = 21$.

$$\begin{array}{rrrr} & & 43 & 37 \\ & 37 & 6 & 1 \\ 6 & 1 & 6 & \\ 0 & 6 & & \end{array} \qquad \frac{43}{37} = 1+\frac{1}{6}+\frac{1}{6} \qquad \frac{1}{1} \quad \frac{7}{6} \quad \frac{43}{37}$$

$$x = -21.6+37t = 37t-126,$$

$$y = 21.7-43t = 147-43t,$$

i.e.

$$x = 37t-15, \quad y = 18-43t,$$

where again, t is any integer.

The convergents of a regular continued fraction as best approximations. Suppose we have expanded any real number A as a regular continued fraction. Let us write P_n/Q_n for the nth order convergent of this continued fraction. The following inequality now holds:

$$\left|A-\frac{P_n}{Q_n}\right| < \frac{1}{Q_nQ_{n+1}}.$$

These convergents are best approximations to the number A in the sense that no rational fraction with denominator not exceeding Q_n can differ from A by less than the fraction P_n/Q_n.

The expansion of an irrational number as a non-terminating regular continued fraction. Every real irrational number can be written uniquely as a non-terminating regular continued fraction. Conversely, every non-terminating regular continued fraction (such a fraction is necessarily convergent, by Seidel's convergence test, see § 2, sec. 2) is the expansion of one and only one real irrational number.

Periodic regular continued fractions. The regular continued fraction $q_0+\frac{1}{q_1}+\frac{1}{q_2}+\dots$ is said to be *purely periodic* if the sequence q_0, q_1, q_2, ... of its partial denominators consists of a repetition, with the same period, of the n numbers $q_0, q_1, \dots, q_{n-1}$.

If the repetition starts with some $q_k (k \geqq 1)$ instead of with q_0, the regular continued fraction is described as *mixed periodic.*

Similar concepts of periodic continued fractions may also be introduced for continued fractions of a general type, although regular periodic continued fractions are primarily discussed in number theory.

The expansion of irrational square roots as periodic continued fractions. Every periodic continued fraction (not necessarily regular) represents an irrational square root. An important role is played in number theory by a theorem of Lagrange, which states that *every irrational square root can be expanded as a periodic regular continued fraction.*

For example,

$$\frac{\sqrt{5}+1}{2} = 1+\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\ldots$$

$$\sqrt{13} = 3+\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{6}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{6}\,{}_{+}\ldots.$$

Notice that the expansion of $\sqrt{13}$ as a continued fraction of general type,

$$\sqrt{13} = 3+(\sqrt{13}-3) = 3+\frac{4}{3+\sqrt{13}} = 3+\frac{2}{3}\,{}_{+}\frac{1}{3}\,{}_{+}\frac{1}{3}\,{}_{+}\frac{1}{3}\,{}_{+}\ldots,$$

is obtained more simply and is more rapidly convergent. Thus the expansion of irrational square roots as regular continued fractions is more of theoretical rather than practical interest.

6. Equivalent and corresponding continued fractions

When transforming the power series $A_0+A_1x+A_2x^2+\ldots$ into a continued fraction, two cases can be encountered: (1) the convergents of the continued fraction coincide with the partial sums of the initial power series; (2) the convergents do not coincide with the partial sums of the initial power series. In the first case the continued fraction is described as *equivalent* to the initial series, and in the second case as *corresponding* to the initial series. The expansion of the nth convergent of the corresponding continued fraction as a power series coincides with the initial power series up to and including the term in x^n. It is clear that an equivalent continued fraction is only another form of writing a power series and does not yield new approximate expressions for its sum. As regards convergence, a power or numerical series and the continued fraction corresponding to it can behave differently. They may be both convergent, or both divergent, or one be convergent and the other divergent. The domains of convergence of a power series and the continued fraction corresponding to it may thus be different. For instance, there are power series with radius of convergence equal to zero, which can be transformed into corresponding continued fractions that are convergent in a fairly wide domain.

Equivalent continued fractions can sometimes be readily transformed into corresponding ones by means of the transformation (5.13). Since it is far easier to construct the equivalent rather than the corresponding fraction, the application of transforming equivalent fractions into corresponding ones is of practical interest.

The construction of equivalent fractions. For this purpose it is easiest to employ *Euler's identity*:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + \frac{c_1 x}{1 -} \, \frac{c_2 x}{c_1 + c_2 x -} \, \frac{c_1 c_3 x}{c_2 + c_3 x -} \, \dots \, - \frac{c_{n-2} c_n x}{c_{n-1} + c_n x -} \, \dots \, . \tag{5.16}$$

This identity can also be given the form

$$\sum_{n=0}^{\infty} c_n x^n = \frac{c_0}{1 -} \, \frac{c_1 x}{c_0 + c_1 x -} \, \frac{c_0 c_2 x}{c_1 + c_2 x -} \, \frac{c_1 c_3 x}{c_2 + c_3 x -} \, \dots$$

$$\dots \, - \frac{c_{n-2} c_n x}{c_{n-1} + c_n x -} \, \dots \, . \tag{5.17}$$

EXAMPLE 11.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots =$$

$$= \frac{x}{1 +} \, \frac{x^2}{3 - x^2 +} \, \frac{9x^2}{5 - 3x^2 +} \, \dots + \frac{(2n-1)^2 x^2}{2n + 1 - (2n-1)x^2 +} \, \dots \, .$$

In particular, when $x = 1$ we have,

$$\frac{\pi}{1} = \frac{1}{1 +} \, \frac{1^2}{2 +} \, \frac{3^2}{2 +} \, \frac{5^2}{2 +} \, \frac{7^2}{2 +} \, \dots + \frac{(2n-1)^2}{2 +} \, \dots \, .$$

$$\frac{0}{1} \quad \frac{1}{1} \quad \frac{2}{3} \quad \frac{13}{15} \quad \frac{76}{105} \quad \frac{789}{985}$$

The last expansion was first obtained by Brouncker (1620–1684). This relationship is regarded as historically the first expansion of a transcendental number as a continued fraction.

On applying transformation (5.13) to the expansion of arc tan x as an equivalent continued fraction, we get the continued fraction

$$\begin{array}{llll} \text{arc tan}\, x = x - & \dfrac{5x^3-3x^5}{15+9x^2} & - \quad \dfrac{9x^2}{1} & - \ldots \\[2ex] \dfrac{x}{1} & \dfrac{15x+4x^2+3x^5}{15+9x^2} & \dfrac{15x-5x^3+3x^5}{15} & \end{array}$$

which is no longer equivalent. In particular, when $x = 1$ we have

$$\begin{array}{l} \dfrac{\pi}{4} = 1 - \dfrac{2}{24} - \dfrac{9}{1} - \dfrac{25}{152} - \dfrac{49}{1} - \dfrac{81}{408} - \ldots - \dfrac{(4n-1)^2}{1} - \\[2ex] \qquad \dfrac{1}{1} \quad \dfrac{22}{24} \quad \dfrac{13}{15} \quad \dfrac{1426}{1680} \quad \dfrac{789}{945} \\[2ex] \hfill - \dfrac{(4n+1)^2}{8(8n^2+8n+3)} - \ldots \end{array}$$

7. The formation of corresponding fractions. Viskovatov's method

The terms of the partial quotients of the corresponding continued fraction can be expressed in terms of the coefficients of the terms of the initial power series, although high order determinations appear in the relationships thus obtained. This makes such relationships of little practical use in the majority of cases. It is therefore better in practice to use a method of successively obtaining the terms of the partial quotients of the corresponding continued fraction from the terms of the power series. Such a method was proposed in principle at the beginning of the nineteenth century by the Russian scholar V. Viskovatov. *Viskovatov's method* amounts to the identity,

$$f(x) = \frac{\alpha_{10}+\alpha_{11}x+\alpha_{12}x^2+\alpha_{13}x^3+\ldots}{\alpha_{00}+\alpha_{01}x+\alpha_{02}x^2+\alpha_{03}x^3+\ldots} = \frac{\alpha_{10}}{\alpha_{00}} + \frac{\alpha_{20}x}{\alpha_{10}} + \frac{\alpha_{30}x}{\alpha_{20}} + \ldots,$$

where

$$\alpha_{mn} = \alpha_{m-1,\,0}\,\alpha_{m-2,\,n+1} - \alpha_{m-2,\,0}\,\alpha_{m-1,\,n+1}.$$

Computation of the coefficients α_{mn} may be conveniently set out in accordance with the scheme:

$$\begin{array}{llll}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \dots \\
\alpha_{10} & \alpha_{11} & \alpha_{12} & \dots \\
\alpha_{20} & \alpha_{21} & \alpha_{22} & \dots \\
\alpha_{30} & \alpha_{31} & \alpha_{32} & \dots \\
\dots & \dots & \dots & \dots
\end{array}$$

EXAMPLE 12. Let us expand the expression $(1-x)/(1-5x+6x^2)$ $(x<1/3)$ as a continued fraction. We have

$$\begin{array}{rrr}
1 & -5 & 6 \\
1 & -1 & \\
-4 & 6 & \\
-2 & & \\
-12. & &
\end{array}$$

Consequently,

$$\frac{1-x}{1-5x+6x^2}=\frac{1}{1\,-}\;\frac{4x}{1}\;-\;\frac{2x}{-4}\;-\;\frac{12x}{-2}=$$

$$=\frac{1}{1\,-}\;\frac{4x}{1}\;+\;\frac{x}{2}\;-\;\frac{3x}{1}$$

$$\frac{0}{1}\quad\frac{1}{1}\quad\frac{1}{1-4x}\quad\frac{2+x}{2-7x}\quad\frac{2-2x}{2-10x+12x^2}$$

If, when computing the coefficients α_{mn}, it turns out that $\alpha_{k0}=0$, then the $(k+2)$th row of the scheme is obtained by means of a shift of the $(k+1)$th row one place to the left; the $(k+3)$th row is obtained by combining the $(k+2)$th and the kth rows in accordance with the general rule, the $(k+4)$th row by combining the $(k+3)$th and $(k+2)$th, and so on. The expansion in this case has the form

$$f(x)=\frac{\alpha_{10}}{\alpha_{00}}+\frac{\alpha_{20}x}{\alpha_{10}}+\frac{\alpha_{30}x}{\alpha_{20}}+\dots$$

$$\dots+\frac{\alpha_{k-1,0}x}{\alpha_{k-2,0}}+\frac{\alpha_{k,1}x^2}{\alpha_{k-1,0}}+\frac{\alpha'_{k+1,1}x}{\alpha_{k,1}}+\frac{\alpha'_{k+2,1}x}{\alpha'_{k+1,1}}+\dots.$$

EXAMPLE 13. Let us expand the expression $(1-3x^3)/(1-x^2-4x^4)$ $\left(x^2 < \left(\sqrt{17}-1\right)/8\right)$ as a continued fraction, we have

$$\frac{1-3x^3}{1-x^2-4x^4} = \frac{1}{1} - \frac{x^2}{1} - \frac{3x}{-1} - \frac{5x}{-3} + \frac{25x}{-5} + \frac{300x}{25} - \frac{15x}{1} =$$

$$\begin{array}{cccccc} = \dfrac{1}{1} - & \dfrac{x^2}{1} + & \dfrac{3x}{1} - & \dfrac{5x}{3} + & & \\ \dfrac{0}{1}\ \dfrac{1}{1} & \dfrac{1}{1-x^2} & \dfrac{1+3x}{1+3x-x^2} & \dfrac{3+34x}{3+4x-3x^2+5x^3} & & \end{array}$$

$$\begin{array}{ccc} + \dfrac{5x}{1} - & \dfrac{12x}{5} - & \dfrac{3x}{1} \\ \dfrac{3+9x+15x^2}{3+9x+12x^2} & \dfrac{15+9x+27x^2}{15+9x+12x^2+36x^3-60x^4} & \dfrac{15-45x^3}{15-15x^2-60x^4}\,. \end{array}$$

$$\begin{array}{rrrrr} 1 & 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & & -3 \\ 0 & -1 & 3 & & -4 \\ -1 & 3 & -4 & & \\ -3 & 4 & 3 & & \\ -5 & 15 & & & \\ 25 & -15 & & & \\ 300 & & & & \\ -300.15. & & & & \end{array}$$

8. Appell's method

In 1913 Appell proposed the following method enabling any positive number to be expanded as a continued fraction.

Let N be any positive number. Let a be the square root of N, computed with a deficiency of not more than 1. Thus $N = a^2+R$, where $0 < R < 2a+1$. We shall assume $a = 0$ if $N < 1$. We put $R = (2a+1)/N_1$, where $N_1 > 1$. Let a_1 be the square root of N_1, computed with deficiency of not more than 1. Then $N_1 = a_1^2+R_1$, where $0 < R_1 < 2a_1+1$. We put $R_1 = (2a_1+1)/N_2$ and continued this process indefinitely. Generally speaking, N is now expanded as a non-terminating continued fraction.

If $N_p = N$, N is expanded a *pure periodic continued fraction.* If $N_p = N_m$ $(1 \leqq m < p)$, N is expanded as *a mixed periodic continued fraction.*

EXAMPLE 14.

$$5 = 2^2+1 = 2^2+\frac{5}{5}, \quad N_1 = N, \quad 5 = 2^2+\frac{5}{2^2+}\,\frac{5}{2^2+}\dots$$

EXAMPLE 15.

$$2 = 1^2+\frac{3}{3}, \quad 3 = 1^2+\frac{3}{\frac{3}{2}}, \quad \frac{3}{2} = 1^2+\frac{3}{6},$$

$$6 = 2^2+\frac{5}{\frac{5}{2}}, \quad \frac{5}{2} = 1^2+\frac{3}{2}; \quad N_5 = N.$$

$$2 = 1^2+\frac{3}{1^2+}\,\frac{3}{1^2+}\,\frac{3}{2^2+}\,\frac{5}{1^2+}\,\Bigg(\frac{3}{1^2+}\,\frac{3}{1^2+}\dots \text{second period} \quad .$$

EXAMPLE 16.

$$\frac{1+\sqrt{13}}{2} = 1^2+\frac{3}{1^2+}\,\frac{3}{1^2+}\,\frac{3}{1^2+}\dots.$$

Appell also pointed out the more general relationship

$$N = a^n+\frac{(a+1)^n-a^n}{a_1^2\quad +}\,\frac{(a_1+1)^n-a_1^n}{a_2^2\quad +\dots}.$$

Here a_1, $a_2, \dots$ are positive integers, while a may also be equal to zero. Here we can take the nth root of the number with a deficiency or with an excess of not more than 1, or we can even alternate in various ways the approximation of the root with a deficiency and an excess in the expansion obtained.

§ 2. Fundamental tests for the convergence of continued fractions

1. The convergence of continued fractions

It was mentioned above that a continued fraction for which $\lim_{n\to\infty} P_n/Q_n$ exists and is finite is described as *convergent*. The value of the continued fraction is in this case taken equal to this limit. But it does not follow from the convergence of a continued fraction that $\lim_{n\to\infty} P_n/Q_n$ is equal to the magnitude which has been expanded as the continued fraction.

Essentially and non-essentially divergent continued fractions. If $\lim_{n\to\infty} P_n/Q_n = +\infty$ or $\lim_{n\to\infty} P_n/Q_n = -\infty$, the continued fraction is said to be *non-essentially divergent*, whereas if $\lim_{n\to\infty} P_n/Q_n$ does not exist, the continued fraction is said to be *essentially divergent*. The concepts of essential and non-essential divergence were introduced by *Perron*.

Unconditionally and conditionally convergent continued fractions.

Unconditionally and conditionally convergent continued fractions.
It is well known that the convergence of a series or infinite product is not affected by the removal of a finite set of terms. In continued fractions, however, the removal of a finite set of partial quotients (excluding the zero partial quotient) can turn a convergent fraction into a non-essentially divergent fraction. Pringsheim therefore introduced the following concepts: a continued fraction $[a_\nu/b_\nu]_1^\infty$ is said to be *unconditionally convergent* if, for all $m \geqq 1$, the fraction $[a_\nu/b_\nu]_m^\infty$ is convergent. Whereas if the latter fraction is divergent for at least one value of m, the fraction $[a_\nu/b_\nu]_1^\infty$ is said to be *conditionally convergent*. It follows from this that it is not in general possible to quote convergence tests for continued fractions in the limiting form as used for series. The convergence conditions, connecting, say a_n and b_n, must be satisfied for all positive integral n. It must be emphasized that the removal of a finite set of partial quotients can turn a convergent continued fraction *only* into a convergent or a non-essentially divergent fraction.

A necessary test for the convergence of a continued fraction $[1/\alpha_\nu]_1^\infty$.

KOCH'S THEOREM. *The convergence of the series* $\sum_{n=1}^{\infty} |\alpha_n|$ *is sufficient for the limits* $\lim_{n\to\infty} P_{2n} = P$, $\lim_{n\to\infty} P_{2n+1} = P'$, $\lim_{n\to\infty} Q_{2n} = Q$, $\lim_{n\to\infty} Q_{2n+1} = Q'$ *to be finite, and the relationship* $P'Q-PQ' = 1$ *to be satisfied.* It follows from this that the divergence of the series $\sum_{n=1}^{\infty} |\alpha_n|$ is *necessary* for the convergence of the fraction $[1/\alpha_\nu]_1^\infty$.

Uniform convergence of a continued fraction. If the terms of the partial quotients of a continued fraction are functions of a finite or infinite set of variables, the continued fraction is said to be *uniformly convergent* on the set E of variation of these variables when its convergents P_n/Q_n tend uniformly to a limit in E, i.e. when, given any $\varepsilon > 0$, we can find a number N such that, for $n \geqq N$, Q_n is non-zero *on the whole* set E and the following inequality holds:

$$\left| \frac{P_n}{Q_n} - \lim_{\lambda\to\infty} \frac{P_\lambda}{Q_\lambda} \right| < \varepsilon. \tag{5.18}$$

It follows from this definition that the series (see [3]):

$$\frac{P_N}{Q_N} + \sum_{\lambda=N+1}^{\infty} \left(\frac{P_\lambda}{Q_\lambda} - \frac{P_{\lambda-1}}{Q_{\lambda-1}} \right) \equiv \frac{P_N}{Q_N} + \sum_{\lambda=N+1}^{\infty} (-1)^{\lambda-1} \frac{a_1 a_2 \ldots a_\lambda}{Q_{\lambda-1} Q_\lambda} \tag{5.19}$$

is now uniformly convergent in E to $\lim_{n\to\infty} P_n/Q_n$, since P_n/Q_n is its partial sum, while $\lim_{n\to\infty} P_n/Q_n$ is its sum. Conversely, condition (5.18) follows from the uniform convergence of this series, i.e. the uniform convergence of the continued fraction. It also follows from this definition that the values of the continued fraction and of the series (5.19) coincide for any $x \in E$, i.e. that the continued fraction and the series (5.19) are identically equal on the set E.

A condition for the convergence of a continued fraction to the function which can be expanded as this continued fraction. The uniform convergence of the continued fraction

$$\frac{c_1}{1} + \frac{c_2 x}{1} + \ldots + \frac{c_n x}{1} + \ldots \quad (c_n \neq 0;\ n = 1, 2, \ldots) \tag{5.20}$$

on the set E is a sufficient condition for this fraction to be convergent on the set E to the function $K(x)$, which can be expanded as the continued fraction.

It follows from this theorem that, if the fraction (5.20) is uniformly convergent for $|x| < \varrho$, it is convergent for $|x| < \varrho$ to a regular single-valued analytic function, which can be expanded as the fraction and which can be expanded as the power series corresponding to the fraction (5.20), this series being convergent to the same function for $|x| < \varrho$. The condition is therefore obtained in order that the continued fraction and the power series which correspond to each other are convergent to the same function. Notice that the domain $|x| < \varrho$ can be replaced by any domain T containing zero as an interior point.

If zero is a boundary point of the set E, then $\varrho = 0$, i.e. the power series corresponding to the fraction (5.20) is divergent everywhere except at zero. But the fraction (5.20), uniformly convergent on the set E, is nevertheless convergent on this set to the function that we have expanded as this continued fraction. This function can be expanded in the present case in a neighbourhood of zero as a divergent power series, i.e. it is not analytic. The convergents of the fraction (5.20) are thus approximate expressions for a non-analytic function, i.e. we have to some degree solved the problem of the approximate evaluation of non-analytic functions.

The condition for identical equality of two uniformly convergent continued fractions. If the values of two continued fractions, uniformly convergent inside a domain T containing zero as an interior point, are coincident in a domain S which is contained wholly in T, then these fractions are identically equal inside the domain T.

Uniform convergence of the fraction $[c_\nu x/1]_{m+1}^{\infty}$. The uniform convergence of the fraction $[c_\nu x/1]_{m+1}^{\infty}$ inside a domain T containing zero is sufficient for the uniform convergence inside T of the fraction $K(x) = [c_1/1;\ c_\nu x/1]_2^{\infty}$, after the possible exclusion of a finite number of points x' of non-essential divergence. The fraction $[c_1/1;\ c_\nu x/1]_2^{\infty}$ is convergent here inside T to a single-valued analytic function $K(x)$, regular inside T, except at points x', which are the poles of this function.

2. A necessary and sufficient test for the convergence of a continued fraction in which the partial quotients have positive terms (Seidel's test)

As has been shown, the divergence of the series $\sum_{n=1}^{\infty} |\alpha_n|$ is *necessary* for the convergence of the fraction $[1/\alpha_n]_1^{\infty}$. Seidel, and independently Stern, showed that if all the terms of the partial quotients of the fraction are positive, the divergence of the series is *necessary and sufficient* for convergence.

Example 17. The continued fraction

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots$$

is convergent by virtue of Siedel's test, since the series 2+2+ . . . is divergent.

Notice that the convergence of a continued fraction in which the partial quotients have positive terms depends on the divergence of a certain series, i.e. the behaviour of the entire set of terms of the partial quotients, not on the behaviour of each of them. A convergent continued fraction in which the partial quotients have positive terms is therefore unconditionally convergent.

On passing from a continued fraction of the form (5.6) to a continued fraction of the general form (5.1), the following statement of Seidel's test is obtained; this statement was proposed by Stern.

The divergence of at least one of the series

$$\sum_{n=1}^{\infty} \frac{a_1 a_3 \dots a_{2n-1}}{a_2 a_4 \dots a_{2n}} b_{2n}, \quad \sum_{n=1}^{\infty} \frac{a_2 a_4 \dots a_{2n}}{a_1 a_3 \dots a_{2n+1}} b_{2n+1} \tag{5.21}$$

is necessary and sufficient for the convergence of the continued fraction (5.1), *of which all the terms of the partial quotients are positive.*

3. Tests sufficent for the convergence of continued fractions in which the partial quotients have positive terms

Establishing the divergence of one of the series (5.21) is usually fairly difficult in practice. Thus it is often more convenient to make use of various tests which are sufficient for the convergence of continued fractions with partial quotients having exclusively positive terms. Some of these tests are given below.

1°. *Divergence of the series* $\sum_{n=2}^{\infty} (b_{n-1}b_n)/a_n$ *is sufficient for the convergence of the fraction* (5.1) *when the partial quotients have positive terms (Stolz).*

2°. *Divergence of the series* $\sum_{n=2}^{\infty} \sqrt{(b_{n-1}b_n)/a_n}$ *is sufficient for the convergence of the fraction* (5.1) *when the partial quotients have positive terms (Saalschütz and Pringsheim).*

3°. *Divergence of the series*

$$\sum_{n=2}^{\infty} \frac{1}{1+\dfrac{a_n}{b_{n-1}b_n}}$$

is sufficient for the convergence of the fraction (5.1) *when the partial quotients have positive terms (Arndt).*

Notice that, if

$$K = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} + \dots,$$

then

$$-K = -b_0 + \frac{a_1}{-b_1} + \frac{a_2}{-b_2} + \dots + \frac{a_n}{-b_n} + \dots.$$

Hence all the tests for convergence of continued fractions in which the partial quotients have exclusively positive terms may readily be extended to continued fractions in which all the partial numerators are positive and all the partial denominators negative.

4. First set of tests sufficient for convergence

Let the numbers $r_1, r_2, \dots$ satisfy the following conditions (Scott and Wall),

$$\left.\begin{array}{ll}
1)\ r_1\,|1+c_2| \geqq |c_2|, & \\
2)\ r_2\,|1+c_2+c_3| \geqq |c_3|, & \\
3)\ r_n\,|1+c_n+c_{n+1}| \geqq r_n r_{n-2}\,|c_n|+|c_{n+1}| & (n \geqq 3), \\
4)\ r_n \gtreqless 0 \quad (n \geqq 3), &
\end{array}\right\} \quad (5.22)$$

where $c_2, c_3, \ldots$ are the partial numerators of the fraction

$$K = \frac{1}{1} + \frac{c_2}{1} + \frac{c_3}{1} + \ldots + \frac{c_n}{1} + \ldots \tag{5.23}$$

If the series $\sum_{n=1}^{\infty} r_1 r_2 \ldots r_n$ *is convergent, then the fraction* (5.23) *is convergent, while*

$$|K| \leqq 1 + \sum_{n=1}^{\infty} r_1 r_2 \ldots r_n. \tag{5.24}$$

If the numbers $r_1, r_2, \ldots$ satisfy conditions (5.22) and at least one of the c_n is zero, the fraction (5.23) is convergent.

If $c_1, c_2, \ldots$ are functions of several variables, the above convergence test takes the following form. *If the numbers* $r_1, r_2, \ldots$ *satisfy conditions* (5.22), *and the series* $\sum_{n=1}^{\infty} r_1 r_2 \ldots r_n$ *is convergent on a set* E, *the fraction* (5.23) *is uniformly convergent on* E, *inequality* (5.24) *being satisfied.*

In particular, the set E can be chosen on the basis of the following theorem due to Scott and Wall:

The set of conditions

$$p_1 > 1, \quad |c_n| \leqq \frac{p_n - 1}{p_{n-1} p_n} \qquad (n = 2, 3, \ldots) \tag{5.25}$$

where p_1, p_2 *are the terms of some numerical sequence, is sufficient for uniform convergence of the fraction* (5.23) *and for the following inequality to be satisfied:*

$$|K| \leqq \frac{p_1}{p_1 - 1}\left[1 - \frac{1}{1 + \sum\limits_{n=1}^{\infty} (p_1 - 1)(p_2 - 1) \ldots (p_n - 1)}\right]. \tag{5.26}$$

Inequality (5.26) becomes equality if $c_n = (1 - p_n)/p_{n-1} p_n$ ($n = 2, 3, \ldots$).

If $p_1 = 1$, Scott and Wall's theorem takes the following form.

The following conditions:

(1) $p_1 = 1$;

(2) $$|c_n| \leqq \frac{p_n - 1}{p_{n-1}p_n} \quad \cdot \ (n = 2, 3, \ldots)$$

(3) *the series* $\sum\limits_{n=1}^{\infty} (p_2 - 1)(p_3 - 1) \ldots (p_{n+1} - 1)$ *is convergent; are sufficient for the uniform convergence of the fraction* (5.23) *and for fulfillment of the inequality*

$$|K| \leqq 1 + \sum_{n=1}^{\infty} (p_2 - 1)(p_3 - 1) \ldots (p_{n+1} - 1).$$

This becomes equality if $c_n = (1 - p_n)/p_{n-1}p_n$ $(n = 2, 3, \ldots)$. In particular, of $p_n = (2n+1)/(n+k)$, the following theorem is obtained, by virtue of (5.25) and (5.26).

The condition

$$|c_n| \leqq \frac{n^2 - (k-1)^2}{4n^2 - 1} \qquad (n = 2, 3, \ldots) \tag{5.27}$$

is sufficient for the uniform convergence of the fraction (5.20), *while we have: when* $k \neq 2$,

$$|K| \leqq \frac{3}{2-k}\left[1 - \frac{1}{1 + \sum\limits_{n=1}^{\infty} \frac{2-k}{1=k}\,\frac{3-k}{2+k} \cdots \frac{n+1-k}{n+k}}\right]. \tag{5.28}$$

An investigation of the series in the denominator on the right-hand side of the last inequality shows that the series is convergent for $k > 1$ and divergent for $k \leqq 1$. In addition, it follows from inequality (5.27) that $-1 < k < 3$. Hence, when $-1 < k \leqq 1$ (5.28) takes the form

$$|K| \leqq \frac{3}{2-k} \tag{5.29}$$

and only retains the form (5.28) for $1 < k < 3$.

Vorpitskii's test. When $k = 1/2$, we have $p_n = 2$. In this case (5.27) takes the form

$$|c_n| \leqq \frac{1}{4} \qquad (n = 2, 3, \ldots), \tag{5.30}$$

while $|K| \leqq 2$ in accordance with (5.29). But we can replace (5.29) by the stricter inequality:

$$\left| K - \frac{4}{3} \right| \leqq \frac{2}{3}. \tag{5.31}$$

VAN VLECK'S THEOREM. *The condition*

$$0 \leqq c_n \leqq g \qquad (n = 2, 3, \ldots) \tag{5.32}$$

is sufficient for the fraction $K = [1/1,\ c_n z/1]_2^\infty$ *to be convergent in the circle* $|z| < 1/4g$ *to a regular analytic non-rational function, which is also equal to the series corresponding to this fraction, while* $|K - 4/3| \leqq 2/3$.

The proof of this theorem follows from Vorpitskii's test.

5. Tests for the convergence of continued fractions periodic in the limit

The fraction $[a_\iota/b_\iota]_1^\infty$, for which $a_\nu \neq 0, \lim\limits_{\nu \to \infty} a_\nu = a, \lim\limits_{\nu \to \infty} b_\nu = b$, is described as *periodic in the limit*. Such fractions are of great practical importance, since they are connected with the expansions as continued fractions of a large number of commonly encountered functions.

The convergence tests for continued fractions periodic in the limit follow from the theorem:

The condition $\limsup\limits_{\nu \to \infty} \{|c_\nu|\} \leqq g$ *is sufficient for the fraction*

$$\left[\frac{c_1}{1};\ \frac{c_\nu z}{1} \right]_2^\infty \tag{5.33}$$

to be convergent in the circle $|z| < 1/4g$ *(excluding possible poles) to a regular analytic non-rational function, the poles of the latter being the points of non-essential divergence of the fraction. In a neighbourhood of zero this function is equal to the series corresponding to the continued fraction* (5.33).

Having chosen as the g in this theorem any positive number, as small as desired, we pass to the following theorem.

The condition $\lim\limits_{\nu \to \infty} c \ = 0$ *is sufficient for the fraction* (5.33) *to be* *uniformly convergent in any finite domain of the complex plane, excluding a finite set of points of non-essential divergence, to an analytic function, which is regular in a neighbourhood of zero, and is regular in the remainder of the domain except for the above-mentoined points of non-essential divergence of the fraction, these being the poles of the function. The point* $z = \infty$ *is essentially a singular point of the function.*

To generalize this theorem, we assume that $\lim\limits_{\nu \to \infty} c_\nu = c \neq 0$. We thus arrive at the following theorem.

The condition $\lim\limits_{\nu \to \infty} c_\nu = c \neq 0$ *is sufficient for the uniform convergence of the fraction* (5.33), *excluding a finite set of points of non-essential divergence in any domain* $T \subset \bar{T}$, *where* $\bar{T}$ *is the complex plane with a cut along the real axis from the point* $(-1/4c, 0)$ *to an infinitely remote point, such that the cut does not pass through zero.*

If the cut starts from zero, the following *Stieltjes test* holds. *Suppose the following conditions are satisfied:*

(1) $\alpha_1, \alpha_2, \ldots$ *are real and non-negative*;

(2) $\alpha_1, \alpha_3, \ldots, \alpha_{2n+1}, \ldots$ *are not all zero*;

(3) *the series* $\sum\limits_{k=1}^{\infty} \alpha_k$ *is divergent.*

Then the fraction $[z/\alpha_k]_1^\infty$ *is uniformly convergent in any finite domain lying inside the complex plane, cut along the negative part of the real axis.*

§ 3. The expansion of certain functions as continued fractions

1. Lagrange's method

Lagrange proposed the following method of solving differential equations with the aid of continued fractions. Suppose we have a differential equation in y and x, and assume that $y \approx \xi_0$, when $|x| \approx 0$. We now put $y = \xi_0/(1+y_1)$ and substitute this relationship into the original equation. We obtain a differential equation in y_1 and x. Suppose $y_1 \approx \xi_1$ when $|x| \approx 0$. We put $y_1 = \xi_1/(1+y_2)$ and repeat the process. All in all, we arrive at the solution of the original equation in the form of the continued fraction $[\xi_n/1]_1^\infty$; ξ_n is more conveniently sought in the form $a_n x^{\nu_n}$, where $\nu_n \geqq 0$.

Many differential equations can be solved by this means, but it generally proves difficult to find the dependence of ξ_n on the index n, i.e. to find the general partial quotient of the continued fraction.

2. Fundamental differential equation

Let us apply Lagrange's method to the equation

$$(\alpha+\alpha' x)xy' + (\beta+\beta' x)y + \gamma y^2 = \delta x, \qquad y(0) = 0. \tag{5.34}$$

Having put $y = \delta x/(\alpha+\beta+y_1)$, the equation is reduced to the form·

$$(\alpha+\alpha' x)xy' + [\alpha+\beta-(\alpha'+\beta')x]y_1 + y_1^2 = [(\alpha+\beta)(\alpha'+\beta')+\gamma\delta]x.$$

On repeating this process, we get the continued fraction

$$y = \frac{\delta x}{\alpha+\beta+} \, \frac{[(\alpha+\beta)(\alpha'+\beta')+\gamma\delta]x}{2\alpha+\beta} \, + \, \frac{(\alpha\alpha'-\alpha\beta'+\alpha'\beta+\gamma\delta)x}{3\alpha+\beta} \, + \ldots$$
$$\ldots + \frac{[(n\alpha+\beta)(n\alpha'+\beta')+\gamma\delta)]x}{2n\alpha+\beta} \, + \, \frac{(n^2\alpha\alpha'-n\alpha\beta'+n\alpha'\beta+\gamma\delta)x}{(2n+1)\alpha+\beta} \, + \ldots.$$

It is readily seen from this that the differential equation

$$(\alpha+\alpha' x^k)xy' + (\beta+\beta' x^k)y + \gamma y^2 = \delta x^k, \qquad y(0) = 0 \tag{5.35}$$

has the solution

$$y = \frac{\delta x^k}{k\alpha+\beta+} \, \frac{[(k\alpha+\beta)(k\alpha'+\beta')+\gamma\delta]x^k}{2k\alpha+\beta} \, +$$
$$+ \, \frac{(k^2\alpha\alpha'-k\alpha\beta'+k\alpha'\beta+\gamma\delta)x^k}{3k\alpha+\beta} \, + \ldots$$
$$\ldots + \frac{[(nk\alpha+\beta)(nk\alpha'+\beta')+\gamma\delta]x^k}{2nk\alpha+\beta} \, +$$
$$+ \, \frac{(n^2k^2\alpha\alpha'-nk\alpha\beta'+nk\alpha'\beta+\gamma\delta)x^k}{(2n+1)k\alpha+\beta} \, + \ldots. \tag{5.36}$$

Almost all the differential equations, the solutions of which were expanded as continued fractions by Lagrange, Euler and others, are particular cases of equation (5.35). It is therefore natural to refer to (5.35) as the *fundamental differential equation.*

3. The expansion of a power function as a continued fraction

Let us put $y = (1+x)^{\nu}$, where ν is any real number. Then $(1+x)y' = {}$ $= \nu y = \nu y$, $y(0) = 1$. On putting $y = 1+\nu x/(1+z)$, we reduce the differential equation to the form

$$(1+x)xz'+[1-(1-\nu)x]z+z^2 = (1-\nu)x, \qquad z(0) = 0.$$

This is a particular case of equation (5.35), in which

$$k = \alpha = \alpha' = \beta = \gamma = 1, \quad \beta' = -(1-\nu), \quad \delta = 1-\nu.$$

We thus have, from expansion (5.36):

$$(1+x)^{\nu} = 1+\frac{\nu x}{1\,+}\,\frac{(1-\nu)x}{2\,+}\,\frac{(1+\nu)x}{3\,+}\,+$$
$$+\frac{(2-\nu)x}{2}+\dots+\frac{(n-\nu)x}{2\,+}\,\frac{(n+\nu)x}{2n+1\,+}\,\dots\,. \tag{5.37}$$

This expansion was obtained by Lagrange. In view of the above-mentioned convergence tests, it is convergent on the complex plane cut along the real axis from $x = -\infty$ to $x = -1$.

The power series into which the function $y = (1+x)^{\nu}$ can be expanded is convergent in an open circle of radius 1 with centre at the origin. The expansion of the function $y = (1+x)^{\nu}$ as a continued fraction is therefore convergent in a much wider domain than the expansion of the same function as a power series.

EXAMPLE 18. On putting $x = 1$, $\nu = 1/3$, we have

$$\sqrt[3]{2} = 1+\frac{1}{3\,+}\,\frac{2}{2\,+}\,\frac{4}{9\,+}\,\frac{5}{2\,+}\dots+\frac{3n-1}{2\,+}\,\frac{3n+1}{3(2n+1)\,+}\dots\,.$$

$$\frac{1}{1}\quad \frac{4}{3}\quad \frac{10}{8}\quad \frac{106}{84}$$

On contracting the expansion (5.37) in accordance with formula (5.9), we get

$$(1+x)^{\nu} = 1+\frac{2\nu x}{2+(1-\nu)x}\,-\,\frac{(1-\nu^2)x^2}{3(2+x)}\,-\,\frac{(4-\nu^2)x^2}{5(2+x)}\,-\,\dots\,-\,\frac{(n^2-\nu^2)x^2}{(2n+1)(2+x)}\,-\,\dots\,.$$

This expansion is convergent in the same domain as the expansion (5.37).

Irrational square roots may be expanded as continued fractions by means of the following elementary method.

Let $\sqrt{x} \approx a$. Then $\sqrt{x} = a+\left(\sqrt{x}-a\right) = a+(x-a^2)/\left(a+\sqrt{x}\right)$. Consequently,

$$\sqrt{x} = a+\frac{x-a^2}{2a}\,+\,\frac{x-a^2}{2a}\,+\,\dots\,.$$

This expansion is convergent on the complex plane, except the negative part of the real axis.

4. The expansion of a logarithmic function as a continued fraction

Expansion (5.37) can be put in the form

$$\frac{(1+x)^{\nu}-1}{\nu} = \frac{x}{1}\,+\,\frac{(1-\nu)x}{2}\,+\,\frac{(1+\nu)x}{3}\,+\,\frac{(2-\nu)x}{2}\,+\,\frac{(2+\nu)x}{5}\,+\,\dots\,+\,\frac{(n-\nu)x}{2}\,+\,\frac{(n+\nu)x}{2n+1}\,+\,\dots\,.$$

On setting $\nu = 0$, we obtain, since $\lim\limits_{\nu\to+\infty} [(1+x)^{\nu}-1]/\nu = \ln(1+x)$:

$$\ln(1+x) = \frac{x}{1}\,+\,\frac{x}{2}\,+\,\frac{x}{3}\,+\,\frac{2x}{2}\,+\,\frac{2x}{5}\,+\,\dots\,+\,\frac{nx}{2}\,+\,\frac{nx}{2n+1}\,+\,\dots\,. \tag{5.38}$$

This expansion and the method of obtaining it from expansion (5.37) were found by Lagrange. Expansion (5.38) is convergent on the complex plane cut along the real axis from $x = -\infty$ to $x = -1$.

EXAMPLE 19. On putting $x = 1$, we have:

$$\ln 2 = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\dots+\frac{n}{2+}\,\frac{n}{2n+1+}\,\dots\,.$$

$$\frac{0}{1}\;\frac{1}{1}\;\;\frac{2}{3}\;\;\frac{7}{10}$$

5. The expansion of an exponential function as a continued fraction

On replacing x by x/v, expansion (5.37) takes the form

$$\left(1+\frac{x}{v}\right)^{v} = 1+\frac{x}{1+}\,\frac{\frac{1-v}{v}x}{2}+\frac{\frac{1+v}{v}x}{3}+\frac{\frac{2-v}{v}x}{2}+$$
$$+\frac{\frac{2+v}{v}x}{5}+\dots+\frac{\frac{n-v}{v}x}{2}+\frac{\frac{n+v}{v}x}{2n+1}+\dots\,.$$

On letting v tend to ∞, we obtain in the limit:

$$e^{x} = 1+\frac{x}{1-}\,\frac{x}{2+}\,\frac{x}{3-}\,\frac{x}{2+}\,\frac{x}{5-}\,\dots-\frac{x}{2+}\,\frac{x}{2n+1-}\,\dots\,. \tag{5.39}$$

This expansion is convergent on the entire complex plane. The expansion and the method of obtaining it from expansion (5.37) were discovered by Lagrange. On contracting the continued fraction (5.39) in accordance with formula (5.9), we arrive at the expansion obtained by Euler:

$$e^{x} = 1+\frac{2x}{2-x+}\,\frac{x^{2}}{6+}\,\frac{x^{2}}{10+}\,\dots+\frac{x^{2}}{2+(2n+1)+}\,\dots\,.$$

This expansion is also convergent on the entire complex plane.

6. Expansion of the function $y=\operatorname{arc}\tan x$ as a continued fraction

The differential equation for this function has the form

$$y' = \frac{1}{1+x^{2}}, \quad y(0) = 0.$$

On setting $y = x/(1+z)$, this equation reduces to the form

$$(1+x^2)xz'+(1-x^2)z+z^2 = x^2.$$

This is a particular case of equation (5.35), in which

$$\alpha = \alpha' = \beta = \gamma = \delta = 1, \quad \beta' = -1, \quad k = 2.$$

We therefore have, from expansion (5·36):

$$\operatorname{arc\,tan} x = \frac{x}{1+z} = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{4x^2}{5+}\,\frac{9x^2}{7+}\cdots+\frac{n^2x^2}{2n+1+}\cdots. \tag{5.40}$$

This expansion was discovered by Lambert (1728–1777). It is convergent on the whole complex plane except for two cuts along the imaginary axis from $-\infty\, i$ to $-i$, and from i to $+\infty\, i$. This expansion is thus convergent in a far wider domain than the power series into which the function $y = \operatorname{arc\,tan} x$ may be expanded (this latter series is convergent inside the unit circle with centre at the origin).

The terms of the partial quotients of fraction (5.40) satisfy conditions (5.14). We can thus obtain a further expansion of arc tan x on the basis of equation (5.15):

$$\operatorname{arc\,tan} x = x-\frac{x^3}{3+}\,\frac{9x^2}{5+}\,\frac{4x^2}{7+}\cdots+\frac{(2n+1)^2x^2}{4n+1}+\frac{(2n)^2x^2}{4n+3}+\cdots.$$

7. Expansion of the function $y = \int_0^x dt/(1+t^k)$ as a continued fraction

The differential equation for this function has the form

$$y' = \frac{1}{1+x^k}, \quad y(0) = 0.$$

On putting

$$y = \frac{x}{1+z},$$

the equation reduces to the form

$$(1+x^k)xz'+(1-x^k)z+z^2 = x^k.$$

This is the particular case of equation (5.35) when $\alpha = \alpha' = \beta = \gamma = \delta = 1$, $\beta' = -1$, k is any number. We therefore obtain, from expansion (5.36):

$$\int_0^x \frac{dt}{1+t^k} = \frac{x}{1\,+}\,\frac{x^k}{k+1\,+}\,\frac{k^2x^k}{2k+1\,+}\,\frac{(k+1)^2x^k}{3k+1}\,+\dots$$
$$\dots+\frac{n^2k^2x^k}{2nk+1\,+}\,\frac{(nk+1)^2x^k}{(2n+1)k+1}\,+\dots. \tag{5.41}$$

This expansion was obtained by Lagrange. It is convergent on the plane of the complex variable x^k, cut along the real axis from $-\infty$ to -1. When $k = 1$, expansion (5.41) becomes the expansion for $\ln(1+x)$; when $k = 2$, it becomes the expansion for arc tan x.

On substituting x^{p+1} for x in the integral $\int_0^x dt/(1+t^k)$ and setting $k(p+1) = q$, we obtain from (5.41) the expansion

$$\int_0^x \frac{t^p\,dt}{1+t^q} = \frac{x^{p+1}}{p+1\,+}\,\frac{(p+1)^2x^q}{q+1+p\,+}\,\frac{q^2x^q}{2q+1+p}\,+\dots$$
$$\dots+\frac{n^2q^2x^q}{2nq+1+p\,+}\,\frac{(nq+1+p)^2x^q}{(2n+1)q+1+p}\,+\dots. \tag{5.42}$$

For example,

$$\int_0^x \frac{t^2\,dt}{1+t^4} = \frac{1}{3\,+}\,\frac{9}{7\,+}\,\frac{16}{11\,+}\,\frac{49}{15}\,+\dots+\frac{16n^2}{8n+3\,+}\,\frac{(4n+3)^2}{8n+7}\,+\dots.$$

$$\frac{0}{1}\quad\frac{1}{3}\quad\frac{7}{30}\quad\frac{93}{378}\quad\frac{1738}{7140}$$

On using conditions (5.14) and equation (5.15), we can write (5.24) in the form

$$\int_0^x \frac{t^p\,dt}{1+t^q} = \frac{x^{p+1}}{p+1} - \frac{x^{p+1+q}}{q+1+p\,+}\,\frac{(q+1+p)^2x^q}{2q+1+p}\,+$$
$$+\frac{q^2x^q}{3q+1+p}\,+\dots+\frac{(nq+1+p)^2x^q}{2nq+1+p}\,+\frac{n^2q^2x^q}{(2n+1)q+1+p}\,+\dots.$$

For example

$$\int_0^1 \frac{t^2\,dt}{1+t^4} = \frac{1}{3-}\,\frac{1}{7+}\,\frac{49}{11+}\,\frac{16}{15+}\,\dots+\frac{(4n+3)^2}{8n+3+}\,\frac{16n^2}{8n+7+}\,\dots\,.$$

$$\frac{1}{3}\quad \frac{4}{21}\quad \frac{93}{378}\quad \frac{1459}{6006}$$

8. Expansion of tan x and tanh x as continued fractions

The differential equation for $y = \tan x$ is of the form

$$y' = 1+y^2, \quad y(0) = 0.$$

On putting $y = x/(1+z)$, this equation reduces to

$$xz'+z+z^2 = -x^2.$$

This is the particular case of (5.35) when $\alpha' = \beta' = 0, \alpha = \beta = \gamma = 1$, $\delta = -1$, $k = 2$. We thus obtain from expansion (5.36):

$$\tan x = \frac{x}{1-}\,\frac{x^2}{3-}\,\frac{x^2}{5-}\,\dots-\frac{x^2}{2n+1-}\,\dots\,. \qquad (5.43)$$

This expansion was discovered by Lambert. It is convergent on the whole plane of the complex variable x, except for the points at which $\tan x$ becomes infinite.

On replacing x by x/i in expansion (5.43) and multiplying the resulting equation by i, we obtain

$$\tanh x = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{x^2}{5+}\,\dots+\frac{x^2}{2n+1+}\,\dots\,. \qquad (5.44)$$

9. Expansion of Prima's function as a continued fraction

The integral

$$\int_x^\infty t^{a-1}\,e^{-t}\,dt \qquad (a > 0,\ x > 0)$$

is known as *Prima's function.* Let us introduce the function

$$y = x^{1-a}e^x \int_x^\infty t^{a-1}e^{-t}\,dt,$$

which satisfies the differential equation

$$xy'-(1-a+x)y = -x.$$

On putting $x = 1/t$, $y = 1/(1+u)$, this equation takes the form

$$t^2u'+[1-(1-a)t]u = (1-a)t, \quad u(0) = 0.$$

This is the particular case of equation (5.35) when $\alpha = 0$, $k = \alpha' = \beta = \gamma = 1$, $\beta' = -(1-a)$, $\delta = 1-a$. We thus obtain from expansion (5.36):

$$u = \frac{(1-a)t}{1} \, \frac{t}{+1+} \, \frac{(2-a)t}{1} \, + \ldots + \frac{nt}{1+} \, \frac{(n+1-a)t}{1} \, + \ldots.$$

On returning to the original notation, we get

$$\int_x^\infty t^{a-1}e^{-t}\,dt = \frac{x^a e^{-x}}{x} \, \frac{1-a}{+1} \, \frac{1}{+x+} \, \frac{2-a}{1} \, + \ldots$$

$$\ldots + \frac{n}{x+} \, \frac{n+1-a}{1} \, + \ldots. \tag{5.45}$$

On contracting this expansion on the basis of formula (5.9), we get

$$\int_x^\infty t^{a-1}e^{-t}\,dt = \frac{x^a e^{-x}}{x+1-a} \, \frac{1-a}{-x+3-a} \, \frac{2(2-a)}{-x+5-a} - \ldots$$

$$\ldots - \frac{n(n-a)}{x+2n+1-a} - \ldots. \tag{5.46}$$

We call $\int_{-\infty}^{x} (e^t/t)\,dt$ *the integral exponential function* and denote it by Ei(x) (see (6.393)). We obtain from expansion (5.45) with $a = 0$ and $x < 0$:

$$\mathrm{Ei}(x) = e^x \left(\frac{1}{x-} \, \frac{1}{1-} \, \frac{1}{x-} \ldots - \frac{n}{1-} \, \frac{n}{x-} \ldots \right). \tag{5.47}$$

The function

$$\mathrm{Ei}\,(\ln x) = \int_0^x \frac{dt}{\ln t}$$

is called the *integral logarithm* and is denoted by li(x) (see (6.394)). We obtain from expansion (5.47):

$$\mathrm{li}\,(x) = \frac{x}{\ln x-} \, \frac{1}{1-} \, \frac{1}{\ln x-} \ldots - \frac{n}{1-} \, \frac{n}{\ln x-} \ldots.$$

10. Expansion of the incomplete gamma function as a continued fraction

To obtain an expansion of the incomplete gamma function, i.e. of the integral

$$\int_0^x t^{a-1}e^{-t}\,dt \qquad (a > 0),$$

we introduce the function

$$y = x^a e^x \int_0^x t^{a-1}e^{-t}\,dt \qquad (x > 0),$$

which satisfies the differential equation

$$xy' + (a-x)y - 1 = 0, \quad y(0) = \frac{1}{a}.$$

On putting $y = 1/(a+y_1)$ in this equation, we get

$$xy_1' + (a+x)y_1 + y_1^2 = -ax.$$

This is the particular case of equation (5.35) when $k = \alpha = \beta' = \gamma = 1$, $\alpha' = 0$, $\beta = a$, $\delta = -a$. We thus have from expansion (5.36)

$$\int_0^x t^{a-1}e^{-t}\,dt = \frac{x^a e^{-x}}{a\,-}\,\frac{ax}{1+a+}\,\frac{x}{2+a-}\,\frac{(1+a)x}{3+a}+$$

$$+\frac{2x}{4+a-}\cdots+\frac{nx}{2n+a-}\,\frac{(a+n)x}{2n+1+a+}\cdots.$$

11. Thiele's formula

The Taylor series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

corresponds in the theory of continued fractions to *Thiele's formula*

$$f(x+h) = f(x) + \frac{h}{rf(x)+}\,\frac{h}{2rr_1f(x)+}\,\frac{h}{3rr_2f(x)}+\ldots$$

$$\ldots+\frac{h}{nrr_{n-1}f(x)}+\ldots.$$

Here $r_n f(x)$ is called the *inverse derivative of the n-th order of the function* $f(x)$. The inverse derivatives are defined by the relationships

$$rf(x) = \frac{1}{f'(x)}, \quad r_2 f(x) = f(x) + 2rr_1 f(x),$$

and in general, for $n \geqq 2$,

$$r_n f(x) = r_{n-2} f(x) + nrr_{n-1} f(x).$$

For example,

$$re^x = e^{-x}, \; r_2 e^x = -e^x, \; \ldots, \; r_{2n} e^x = (-1)^n e^x,$$
$$r_{2n+1} e^x = (-1)^n (n+1) e^x.$$

Hence

$$(2n+1) rr_{2n} e^x = (-1)^n (2n+1) e^{-x},$$
$$(2n+2) rr_{2n+1} e^x = 2(-1)^{n+1} e^x.$$

If we use these relationships and replace x by 0 and h by x in Thiele's formula, we again arrive at expansion (5.39).

Thus Thiele's formula is another source of obtaining the expansions of various functions as corresponding continued fractions.

12. Fractional approximations for sin x and sinh x

The general form of the expansion as a continued fraction of sin x is not known. Only a finite set of partial quotients of the expansion can be found by Viskovatov's method. For example,

$$\sin x = \frac{x}{1+}\;\frac{x^2}{6-}\;\frac{7x^2}{10+}\;\frac{11x^2}{98-}\;\frac{551x^2}{198+}\cdots$$
$$\frac{x}{1} \quad \frac{6x}{6+x^2} \quad \frac{60x-7x^3}{60+3x^2} \quad \frac{5880x-620x^3}{5880+360x^2+11x^4}$$

and

$$\sin x = x - \frac{x^3}{6+}\;\frac{3x^2}{10-}\;\frac{11x^2}{42+}\;\frac{25x^2}{66-}\cdots$$
$$\frac{x}{1} \quad \frac{6x-x^3}{6} \quad \frac{60x-7x^3}{60+3x^2} \quad \frac{2520x-360x^3+11x^5}{2520+60x^2}$$

It follows from the approximation

$$\sin x \approx \frac{60x-7x^2}{60+3x^2} \tag{5.48}$$

that $\sin \pi/4 \approx 0{\cdot}7071$, if we put $\pi/4 \approx 0{\cdot}7854$. Hence, in view of the periodicity of the function $y = \sin x$ and the relationships $\sin\left(\frac{1}{2}\pi - x\right) = \cos x$ and $\cos x = \sqrt{1-\sin^2 x}$, the approximation (5.48) enables four-figure tables of this function to be evaluated with the aid of a calculating machine. In the same way, by using the higher-order convergents, we can evaluate tables of $y = \sin x$ with any number of correct decimal places with the aid of a calculating machine.

Since $\sinh x = -i \sin ix$, the expansions obtained for $\sin x$ lead us to the following expansions for $\sinh x$:

$$\sinh x = \frac{x}{1-}\;\frac{x^2}{6}\;+\;\frac{7x^2}{10}\;-\;\frac{11x^2}{98}\;+\;\frac{551x^2}{198}\;-\;\dots,$$

$$\frac{x}{1}\quad \frac{6x}{6-x^2}\quad \frac{60x-7x^3}{60-3x^2}\quad \frac{5880x+620x^3}{5880-360x^2+11x^4}$$

and

$$\sinh x = x + \;\frac{x^3}{6}\;-\;\frac{3x^2}{10}\;+\;\frac{11x^2}{42}\;-\;\frac{25x^2}{66}\;+\;\dots,$$

$$\frac{x}{1}\quad \frac{6x+x^3}{6}\quad \frac{60x+7x^3}{60-2x^2}\quad \frac{2520x+360x^2+11x^5}{2520-60x^2}$$

13. Fractional approximations for cos x and cosh x

The general form of the expansion as a continued fraction of $\cos x$ is unknown. Only a finite set of the partial quotients of the expansion can be obtained by Viskovatov's method. For example,

$$\cos x = \frac{1}{1}+\frac{x^2}{2}-\frac{5x^2}{6}+\frac{3x^2}{50}-\frac{313x^2}{126}+\cdots.$$

$$\frac{1}{1}\qquad \frac{2}{2+x^2}\qquad \frac{12-5x^2}{12+x^2}\qquad \frac{600-244x^2}{600+56x^2+3x^4}$$

and

$$\cos x = 1-\frac{x^2}{2}+\frac{x^2}{6}-\frac{3x^2}{10}+\frac{13x^2}{126}-\cdots.$$

$$\frac{1}{1}\qquad \frac{2-x^2}{2}\qquad \frac{12-5x^2}{12+x^2}\qquad \frac{120-56x^2+3x^4}{120+4x^2}$$

Since $\cosh x = \cos ix$, the above lead to the expansions:

$$\cosh x = \frac{1}{1}-\frac{x^2}{2}+\frac{5x^2}{6}-\frac{3x^2}{50}+\frac{313x^2}{126}-\cdots.$$

$$\frac{1}{1}\qquad \frac{2}{2-x^2}\qquad \frac{12+5x^2}{12-x^2}\qquad \frac{600+244x^2}{600-56x^2+3x^4}$$

and

$$\cosh x = 1+\frac{x^2}{2}-\frac{x^2}{6}+\frac{3x^2}{10}-\frac{13x^2}{126}+\cdots.$$

$$\frac{1}{1}\qquad \frac{2+x^2}{2}\qquad \frac{12+5x^2}{12-x^2}\qquad \frac{120+56x^2+3x^4}{120-4x^2}$$

14. Fractional approximation for the error function

The general form of the expansion as a continued fraction of the error function $\operatorname{erf} x = \left(2/\sqrt{\pi}\right)\int_0^x e^{-t^2}\,dt$ is unknown. Only a finite set of partial quotients of the expansion can be obtained by Viskovatov's method. For instance

$$\int_0^x e^{-t^2}\,dt = \frac{x}{1}+\frac{x^2}{3}-\frac{x^2}{10}+\frac{39x^2}{7}-\frac{739x^2}{234}+\cdots.$$

$$\frac{x}{1}\qquad \frac{3x}{3+x^2}\qquad \frac{30x-x^3}{30+9x^2}\qquad \frac{210x+110x^3}{210+180x^2+39x^4}$$

15. Conversion of Stirling's series into a continued fraction

Let us take the *Stirling* series (see (6.509) and (6.516)):

$$\ln \Gamma(x) = \left(x-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + J(x),$$

where

$$J(x) = \frac{B_2}{1.2x} + \frac{B_4}{3.4x^3} + \dots + \frac{B_{2n}}{(2n-1)2nx^{2n-1}} + \dots,$$

and B_2, B_4, ... are Bernoulli numbers (see Chapter VI, § 2, sec. 1, 1°):

$$B_2 = \frac{1}{6},\quad B_4 = -\frac{1}{30},\quad B_6 = \frac{1}{42},\quad B_8 = -\frac{1}{30},\quad B_{10} = \frac{5}{66},\dots.$$

By using Viskovatov's method, we can convert the Stirling series into a continued fraction, the general partial quotient of which is unknown:

$$\ln \Gamma(x) = \left(x-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi +$$

$$+\frac{1}{12x+}\,\frac{2}{5x+}\,\frac{53}{42x+}\,\frac{1170}{53x+}\,\frac{22999}{429x+}\dots.$$

16. Fractional approximation for the gamma function

If we evaluate the coefficients in the expansion of the gamma function:

$$\Gamma(1+x) = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$c_0 = 1,\quad c_n = \frac{1}{n}\sum_{k=1}^{n}(-1)^k c_{n-k}s_k,\quad s_k = \sum_{m=1}^{\infty}\frac{1}{m^k}\quad (k \geqq 2),$$

$s_1 = C \approx 0{\cdot}57722$ *(Euler's constant*, see (6.56)), we obtain approximately:

$$\Gamma(1+x) \approx 1 - 0{\cdot}57722x + 0{\cdot}98906x^2 - 0{\cdot}90748x^3 + 0{\cdot}98173x^4 - \dots$$

On applying Viskovatov's method to this approximation, we obtain in particular:

$$\Gamma(1+x) \approx \frac{1+0{\cdot}15141x+0{\cdot}24392x^2}{1+0{\cdot}72263x-0{\cdot}32456x^2} = T(x). \tag{5.49}$$

This approximation is sufficiently accurate in many cases, as is clear from the following table:

x	$\Gamma(1+x)$	$T(x)$	x	$\Gamma(1+x)$	$T(x)$
−0·5	1·7725	1·7767	0·1	0·95135	0·95135
−0·4	1·4892	1·4902	0·2	0·91817	0·91816
−0·3	1·29806	1·29823	0·3	0·89747	0·89743
−0·2	1·16423	1·16425	0·4	0·88726	0·88711
−0·1	1·06863	1·06863	0·5	0·8862	0·8858
0	1	1	0·6	0·8935	0·8927
			0·7	0·9086	0·9071
			1	1	0·994

17. Fractional approximation for the logarithm of the gamma function

Approximation of this function on the basis of Stirling's series requires a knowledge of $\ln x$. However, it is possible to obtain a simpler approximation by starting out from the expansion

$$\ln\Gamma(1+x) = -Cx+\sum_{n=2}^{\infty}\frac{(-1)^n x^n}{n}s_n, \tag{5.50}$$

where $s_n = \sum_{m=1}^{\infty} 1/m^n$, C is Euler's constant. On evaluating the coefficients, we obtain approximately:

$$\ln\Gamma(1+x) \approx -0{\cdot}57722x+0{\cdot}82247x^2-0{\cdot}40068x^3+0{\cdot}27058x^4-\dots.$$

On applying Viskovatov's method to this approximation, we obtain in particular:

$$\ln\Gamma(1+x) \approx \frac{-0{\cdot}57722x+0{\cdot}59769x^2}{1+0{\cdot}38941x-0{\cdot}13928x^2}\,.$$

Hence

$$\lg \Gamma(1+x) \approx \frac{-0\cdot25068x+0\cdot25957x^2}{1+0\cdot38941x-0\cdot13928x^2} = T(x). \tag{5.51}$$

The accuracy of this approximation is clear from the table.

x	$\lg \Gamma(1+x)$	$T(x)$	x	$\lg \Gamma(1+x)$	$T(x)$
$-0\cdot5$	$0\cdot2486$	$0\cdot2469$	$0\cdot1$	$\bar{1}\cdot97834$	$\bar{1}\cdot97834$
$-0\cdot4$	$0\cdot1730$	$0\cdot1725$	$0\cdot2$	$\bar{1}\cdot96292$	$\bar{1}\cdot96293$
$-0\cdot3$	$0\cdot11329$	$0\cdot11321$	$0\cdot3$	$\bar{1}\cdot95302$	$\bar{1}\cdot95305$
$-0\cdot2$	$0\cdot066039$	$0\cdot066029$	$0\cdot4$	$\bar{1}\cdot94806$	$\bar{1}\cdot94818$
$-0\cdot1$	$0\cdot0288268$	$0\cdot0288266$	$0\cdot5$	$\bar{1}\cdot9475$	$\bar{1}\cdot9479$
0	0	0			

18. Fractional approximation for the derivative of the logarithm of the gamma function

In accordance with (5.50), the function

$$\Psi(x) = \frac{d}{dx}\ln \Gamma(1+x)$$

has the expansion

$$\Psi(x) = -C+s_2x-s_3x^2+\ldots+(-1)^{n+1}s_{n+1}x^n+\ldots,$$

from which it follows that

$$\Psi(x) \approx -0\cdot57722+1\cdot64493x-1\cdot20206x^2+1\cdot08232x^3-\ldots.$$

On applying Viskovatov's method to the last approximation, we obtain in particular:

$$\Psi(x) \approx \frac{-0\cdot57722+1\cdot25687x}{1+0\cdot67228x-0\cdot16670x^2} = T(x). \tag{5.52}$$

The accuracy of this approximation is clear from the table.

x	$\Psi(x)$	$T(x)$	x	$\Psi(x)$	$T(x)$
−0·5	−1·964	−1·938	0·1	−0·4238	−0·4237
−0·4	−1·541	−1·538	0·2	−0·2889	−0·2890
−0·3	−1·225	−1·218	0·3	−0·1692	−0·1687
−0·2	−0·9650	−0·9647	0·4	−0·0614	−0·0600
−0·1	−0·7549	−0·7549	0·5	0·0365	0·0396
0	−0·57722	−0,57722			

19. Obreshkov's formula

A knowledge of the general form of the partial quotients of the continued fraction into which a given function can be expanded is not sufficient for a determination of the general form of the convergent of this expansion. Nevertheless, the general form of the convergent can be found in certain cases. The most general approach to this problem uses *Obreshkov's formula*, which is one of the generalizations of Taylor's formula. Obreshkov's formula is

$$\sum_{\nu=0}^{k}(-1)^{\nu}\frac{C_k^{\nu}}{C_{m+k}^{\nu}}\frac{(x-x_0)^{\nu}}{\nu!}f^{(\nu)}(x)=\sum_{\nu=0}^{m}\frac{C_m^{\nu}}{C_{m+k}^{\nu}}\frac{(x-x_0)^{\nu}}{\nu!}f^{(\nu)}(x_0)+$$

$$+\frac{1}{(k+m)!}\int_{x_0}^{x}(x-t)^m(x_0-t)^k f^{(m+k+1)}(t)\,dt. \qquad (5.53)$$

Determination of the rational fraction approximations of a power function with the aid of Obreshkov's formula. We put $x_0 = 1, f(x) = x^n$ in (5.53), n being any real number. Now,

$$x^n\approx\frac{\displaystyle\sum_{\nu=0}^{m}\frac{C_m^{\nu}C_n^{\nu}}{C_{m+k}^{\nu}}(x-1)^{\nu}}{\displaystyle\sum_{\nu=0}^{k}(-1)^{\nu}\frac{C_k^{\nu}C_n^{\nu}}{C_{m-k}^{\nu}}\frac{(x-1)^{\nu}}{x^{\nu}}}.$$

When $m = k$, this equation becomes

$$x^n\approx\frac{\displaystyle\sum_{\nu=0}^{k}\frac{C_k^{\nu}C_n^{\nu}}{C_{2k}^{\nu}}(x-1)^{\nu}}{\displaystyle\sum_{\nu=0}^{k}(-1)^{\nu}\frac{C_k^{\nu}C_n^{\nu}}{C_{2k}^{\nu}}\frac{(x-1)^{\nu}}{x^{\nu}}}.$$

In particular, when $k = 1$,

$$x^n \approx \frac{2-n+nx}{n+(2-n)x}\,x.$$

For instance, when $x = 2$ and $n = 1/3$, we have $\sqrt[3]{2} \approx 14/11 \approx 1{\cdot}273$ (the accurate value is $1{\cdot}2599\ldots$).

Determination of the rational approximations of the exponential function with the aid of Obreshkov's formula. We put $x_0 = 0$, $f(x) = e^x$ in (5.53). Now,

$$e^x \approx \frac{\sum\limits_{\nu=0}^{m} \dfrac{C_m^\nu}{C_{m+k}^\nu}\dfrac{x^\nu}{\nu!}}{\sum\limits_{\nu=0}^{k} (-1)^\nu \dfrac{C_k^\nu}{C_{m+k}^\nu}\dfrac{x^\nu}{\nu!}}\,.$$

In particular, when $m = k$,

$$e^x \approx \frac{2k(2k-1)\ldots(k+1)+C_k^1(2k-1)\ldots(k+1)x+C_k^2(2k-2)\ldots(k+1)x^2+\ \ldots x^k}{2k(2k-1)\ldots(k+1)-C_k^1(2k-1)\ldots(k+1)x+C_k^2(2k-2)\ldots(k+1)x^2-\ \ldots+(-1)^k x^k} \tag{5.54}$$

Using the relationship

$$\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$$

together with (5.54), we get

$$\tanh x \approx \frac{C_k^1(2k-1)(2k-2)\ldots(k+1)x-C_k^3(2k-3)(2k-4)\ldots(k+1)4x^3+\ \ldots}{k(2k-1)\ldots(k+1)x+C_k^2(2k-2)(2k-3)\ldots(k+1)2x^2+C_k^4(2k-4)\ldots\ k+1)8x^4+\ldots}$$

On replacing x by ix in the last equation and dividing the right-hand left-hand sides by i, we get the general expression for the convergents of the expansion of $\tan x$ as a continued fraction.

Determination of the rational fraction approximations of the logarithmic function with the aid of Obreshkov's formula. We put $x_0 = 1$, $f(x) = \ln x$ in (5.53). Now,

$$\ln x \approx \sum_{\nu=1}^{m} (-1)^{\nu-1} \frac{C_m^\nu}{C_{m+k}^\nu}\frac{(x-1)^\nu}{\nu} + \sum_{\nu=1}^{k} \frac{C_k^\nu}{C_{m+k}^\nu}\frac{(x-1)^\nu}{\nu x^\nu}\,.$$

In particular, when $m = k$,

$$\ln x \approx \sum_{\nu=1}^{k} \frac{C_k^\nu}{C_{2k}^\nu}\left[(-1)^{\nu-1}+\frac{1}{x^\nu}\right]\frac{(x-1)^\nu}{\nu}\,.$$

For instance, when $k = 1$,

$$\ln x \approx \frac{1}{2}\left(x-\frac{1}{x}\right).$$

When $k = 2$,

$$\ln x \approx \frac{x^2-1}{12x^2}(8x-x^2-1)$$

§ 4. Matrix methods

1. Extraction of the square root by means of second-order matrices

The theory of continued fractions is built up with the aid of the fundamental recurrence relations (5.2). It seems natural to consider generalizations of continued fractions, based on other linear recurrence relations, connecting the numerators and denominators of neighbouring convergents. Let us consider in more detail the relations

$$\left.\begin{aligned} P_n &= \alpha_n P_{n-1}+\beta_n Q_{n-1}, \\ Q_n &= \gamma_n P_{n-1}+\delta_n Q_{n-1}, \end{aligned}\right\} (n = 1, 2, \ldots). \tag{5.55}$$

These equations can be written with the aid of matrices in the form

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}\begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix} \qquad (n = 1, 2, \ldots). \tag{5.56}$$

We put $\alpha_n = a$, $\beta_n = u$, $\gamma_n = 1$ $\delta_n = a$ $(n = 1, 2, \ldots)$. Relations (5.55) and (5.56) now become respectively

$$\left.\begin{aligned} P_n &= aP_{n-1}+uQ_{n-1} \\ Q_n &= P_{n-1}+aQ_{n-1} \end{aligned}\right\} (n = 1, 2, \ldots) \tag{5.57}$$

and

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} a & u \\ 1 & a \end{pmatrix}\begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix} \qquad (n = 1, 2, \ldots). \tag{5.58}$$

When $a = 0$, the process (5.57) is divergent. We shall therefore assume that $a \neq 0$. If $\lim\limits_{n\to\infty} P_n/Q_n$ exists, it can be equal either to $\sqrt{u}$

or to $-\sqrt{u}$. Consequently the process (5.57) is divergent when $a < 0$. Let us use the notation

$$-\sqrt{u} = x_1, \quad \sqrt{u} = x_2.$$

We now have four cases:

$$\left.\begin{array}{l} 1)\ a > x_2,\ x_2 < \dfrac{P_n}{Q_n} < \dfrac{P_{n-1}}{Q_{n-1}}\,; \\ 2)\ 0 < a < x_2; \\ \dfrac{P_{2n-2}}{Q_{2n-2}} < \dfrac{P_{2n}}{Q_{2n}} < x_2 < \dfrac{P_{2n+1}}{Q_{2n+1}} < \dfrac{P_{2n-1}}{Q_{2n-1}}\,; \end{array}\right\} \quad \lim_{n\to\infty} \frac{P_n}{Q_n} = x_2;$$

$$\left.\begin{array}{l} 3)\ x_1 < a < 0, \\ \dfrac{P_{2n-1}}{Q_{2n-1}} < \dfrac{P_{2n+1}}{Q_{2n+1}} < x_1 < \dfrac{P_{2n}}{Q_{2n}} < \dfrac{P_{2n-2}}{Q_{2n-2}}\,; \\ 4)\ a < x_1,\ \dfrac{P_{n-1}}{Q_{n-1}} < \dfrac{P_n}{Q_n} < x_1; \end{array}\right\} \quad \lim_{n\to\infty} \frac{P_n}{Q_n} = x_1.$$

For example:

1) $a > x_2$,

$$\begin{pmatrix} 5 & 22 \\ 1 & 5 \end{pmatrix} \quad \frac{5}{1} \quad \frac{47}{10} \quad \frac{455}{97} \quad \frac{4409}{940}$$

$$5 \quad 4{\cdot}7 \quad 4{\cdot}6907 \quad 4{\cdot}69043$$

$$\sqrt{22} < 4{\cdot}69043;$$

2) $0 < a < x_2$,

$$\begin{pmatrix} 5 & 27 \\ 1 & 5 \end{pmatrix} \quad \frac{5}{1} \quad \frac{52}{10} = \frac{26}{5} \quad \frac{265}{51} \quad \frac{2702}{520} = \frac{1351}{260}$$

$$5 \quad 5{\cdot}2 \quad 5{\cdot}19607 \quad 5{\cdot}19616$$

$$5{\cdot}19607 < \sqrt{27} < 5{\cdot}19616;$$

3) $x_1 < a < 0$,

$$\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \quad \frac{-1}{1} \quad \frac{3}{-2} \quad \frac{-7}{5} \quad \frac{17}{-12} \quad \frac{-41}{29}$$

$$-1 \quad -1{\cdot}5 \quad -1{\cdot}4 \quad -1{\cdot}416 \quad -1{\cdot}4137$$

$$-\ 1{\cdot}416 < -\sqrt{2} < -1{\cdot}4138;$$

4) $a < x_1$,

$$\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix} \quad \frac{-2}{1} \quad \frac{7}{-4} \quad \frac{-26}{15} \quad \frac{97}{-56}$$

$$-2 \quad -1{\cdot}75 \quad -1{\cdot}734 \quad -1{\cdot}73214$$

$$-1{\cdot}73214 < -\sqrt{3}.$$

2. Solution of quadratic equations with the aid of second-order matrices

Let us put $\alpha_n = a$, $\beta_n = -q$, $\gamma_n = 1$, $\delta_n = a+p$ in (5.55). Relationships (5.55) and (5.56) now become

$$\left.\begin{aligned} P_n &= aP_{n-1} - qQ_{n-1} \\ Q_n &= P_{n-1} + (a+P)Q_{n-1} \end{aligned}\right\} \tag{5.59}$$

and

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} a & -q \\ 1 & a+p \end{pmatrix} \begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix}. \tag{5.60}$$

If $\lim\limits_{n\to\infty} P_n/Q_n$ exists, it can only be a root of the quadratic equation $x^2+px+q = 0.$ Consequently, when $p^2-4q < 0$, the process (5.59) is divergent, since the roots of the quadratic equation are complex in this case, while a sequence with real terms cannot have a complex limit.

Let us write

$$x_1 = \frac{-p-\sqrt{p^2-4q}}{2}, \qquad x_2 = \frac{-p+\sqrt{p^2-4q}}{2}.$$

When $a = \frac{1}{2}(x_1+x_2)$, the process (5.59) is divergent. When the process is convergent, we have four cases:

$$\left.\begin{array}{l} 1)\ a > x_2, \quad x_2 < \dfrac{P_n}{Q_n} < \dfrac{P_{n-1}}{Q_{n-1}}\,; \\[2ex] 2)\ \dfrac{x_1+x_2}{2} < a < x_2, \\[2ex] \dfrac{P_{2n-2}}{Q_{2n-2}} < \dfrac{P_{2n}}{Q_{2n}} < x_2 < \dfrac{P_{2n+1}}{Q_{2n+1}} < \dfrac{P_{2n-1}}{Q_{2n-1}}\,; \end{array}\right\} \quad \lim_{n\to\infty} \frac{P_n}{Q_n} = x_2;$$

$$\left.\begin{array}{c} 3)\ x_1 < a < \dfrac{x_1+x_2}{2} \\[2ex] \dfrac{P_{2n-1}}{Q_{2n-1}} < \dfrac{P_{2n+1}}{Q_{2n+1}} < x_1 < \dfrac{P_{2n}}{Q_{2n}} < \dfrac{P_{2n-2}}{Q_{2n-2}}, \\[2ex] 4)\ a < x_1,\ \dfrac{P_{n-1}}{Q_{n-1}} < \dfrac{P_n}{Q_n} < x_1; \end{array}\right\} \quad \lim_{n\to\infty}\frac{P_n}{Q_n} = x_1.$$

For instance, the equation

$$x^2+2x-1 = 0$$

has the roots

$$x_{1,2} = -1\pm\sqrt{2} \quad \text{or} \quad x_1 \approx -2{\cdot}414 \quad \text{and} \quad x_2 \approx 0{\cdot}414.$$

1) $a > x_2$,

$$a = 1,\quad \begin{pmatrix}1&1\\1&3\end{pmatrix} \begin{array}{cccc} \dfrac{1}{1} & \dfrac{2}{4}=\dfrac{1}{2} & \dfrac{3}{7} & \dfrac{10}{24}=\dfrac{5}{12}; \\ 1 & 0{\cdot}5 & 0{\cdot}43 & 0{\cdot}417 \end{array}$$

2) $\dfrac{x_1+x_2}{2} < a < x_2$,

$$a = 0,\quad \begin{pmatrix}0&1\\1&2\end{pmatrix} \begin{array}{cccc} \dfrac{0}{1} & \dfrac{1}{2} & \dfrac{2}{5} & \dfrac{5}{12} \\ 0 & 0{\cdot}5 & 0{\cdot}40 & 0{\cdot}417 \end{array}$$

3) $x_1 < a < \dfrac{x_1+x_2}{2}$.

$$a = -2,\quad \begin{pmatrix}-2&1\\1&0\end{pmatrix} \begin{array}{cccc} \dfrac{-2}{1} & \dfrac{5}{-2} & \dfrac{-12}{5} & \dfrac{29}{-12}; \\ -2 & -2{\cdot}5 & -2{\cdot}40 & -2{\cdot}416 \end{array}$$

4) $a < x_1$,

$$a = -3,\quad \begin{pmatrix}-3&1\\1&-1\end{pmatrix} \begin{array}{cccc} \dfrac{-3}{1} & \dfrac{10}{-4}=\dfrac{5}{-2} & \dfrac{-17}{7} & \dfrac{58}{-24}=\dfrac{29}{-12}. \\ -3 & -2{\cdot}5 & -2{\cdot}428 & -2{\cdot}416 \end{array}$$

3. The connection between matrix methods and the theory of continued fractions

Let us find the conditions under which the equations (5.55) can turn into equations (5.2). For this, we replace n by $n-1$ in equations (5.55), find Q_{n-1} and P_{n-1} from the relationships obtained, and substitute them in equations (5.55). On introducing, for brevity, the notation $\alpha_{n-1}\,\delta_{n-1}-\beta_{n-1}\gamma_{n-1} = \Delta_{n-1}$, we get

$$P_n = \left(\alpha_n+\frac{\beta_n\delta_{n-1}}{\beta_{n-1}}\right)P_{n-1}-\frac{\beta_n\Delta_{n-1}}{\beta_{n-1}}P_{n-2} \qquad (5.61)$$

and

$$Q_n = \left(\frac{\gamma_n}{\gamma_{n-1}}\alpha_{n-1}+\delta_n\right)Q_{n-1}-\frac{\gamma_n\Delta_{n-1}}{\gamma_{n-1}}Q_{n-2}. \qquad (5.62)$$

Equations (5.61) and (5.62) show that P_n and Q_n can be taken as the numerator and denominator of the nth order convergent of a continued fraction provided that

$$\frac{\gamma_n}{\gamma_{n-1}}\alpha_{n-1}+\delta_n = \frac{\beta_n}{\beta_{n-1}}\delta_{n-1}+\alpha_n, \quad \frac{\beta_n}{\beta_{n-1}} = \frac{\gamma_n}{\gamma_{n-1}}. \qquad (n \geqq 2).$$

These conditions are readily written in the form

$$\left.\begin{aligned} \frac{\alpha_n-\delta_n}{\beta_n} &= \frac{\alpha_{n-1}-\delta_{n-1}}{\beta_{n-1}} = \ldots = \frac{\alpha_1-\delta_1}{\beta_1}, \\ \frac{\beta_n}{\gamma_n} &= \frac{\beta_{n-1}}{\gamma_{n-1}} = \ldots = \frac{\beta_1}{\gamma_1}. \end{aligned}\right\} \qquad (5.63)$$

Hence the matrix

$$\begin{pmatrix} \alpha_n & \beta_n \\ \dfrac{\gamma_1}{\beta_1}\beta_n & \alpha_n-\dfrac{\alpha_1-\delta_1}{\beta_1}\beta_n \end{pmatrix} \qquad (5.64)$$

is equivalent to a continued fraction, the partial numerators a_n and partial denominators b_n of which are given by the equations

$$a_n = \frac{\beta_n}{\beta_{n-1}}(\beta_{n-1}\gamma_{n-1}-\alpha_{n-1}\delta_{n-1}), \quad b_n = \frac{\beta_n}{\beta_{n-1}}\delta_{n-1}+\alpha_{n-1} \quad (n \geqq 2).$$

We can find from the relationships

$$\begin{pmatrix}\alpha_0 & \beta_0\\ \gamma_0 & \delta_0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}\alpha_0\\ \gamma_0\end{pmatrix}\equiv\begin{pmatrix}b_0\\ 1\end{pmatrix}$$

and

$$\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta\end{pmatrix}\begin{pmatrix}b_0\\ 1\end{pmatrix}=\begin{pmatrix}\alpha_1 b_0+\beta_1\\ \gamma_1 b_0+\delta_1\end{pmatrix}\equiv\begin{pmatrix}b_0 b_1+a_1\\ b_1\end{pmatrix}$$

the zero and first partial quotients of the fraction. Hence the matrix (5.64), in conjunction with the matrix $\begin{pmatrix}\alpha_0 & \beta_0\\ 1 & \delta_0\end{pmatrix}$, leads to the continued fraction

$$\alpha_0+\frac{\alpha_1\alpha_0+\beta_1-\alpha_0(\gamma_1\alpha_0+\delta_1)}{\gamma_1\alpha_0-\delta_1}\;\frac{\dfrac{\beta_2}{\beta_1}(\beta_1\gamma_1-\alpha_1\delta_1)}{+\;\dfrac{\beta_2}{\beta_1}\delta_1+\alpha_2}\;+\dots$$

$$\dots+\;\frac{\dfrac{\beta_n}{\beta_{n-1}}(\beta_{n-1}\gamma_{n-1}-\alpha_{n-1}\delta_{n-1})}{\dfrac{\beta_n}{\beta_{n-1}}\delta_{n-1}+\alpha_n}\;+\;\dots\;. \qquad (5.65)$$

EXAMPLE 20. The matrix $\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$ leads to $\sqrt{2}$ and is equivalent to the continued fraction

$$\sqrt{2}=1+\frac{1}{2}\,\frac{1}{+2}+\dots\frac{1}{+2}+\dots.$$

In the theory of continued fractions, contraction is performed with the aid of the fairly complicated formula (5.9) and analogous formulae. Here, it is performed much more easily, with the aid of squaring, cubing, etc., the matrix. We have:

$$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}=\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix},\quad a_n=2\cdot 4-3\cdot 3=-1,$$
$$b_n=3+3=6\qquad (n\geqq 2).$$

The zero and first partial quotient of the new continued fraction have to be found directly, and not on the basis of this matrix, from a knowledge of the first convergents of the initial expansion for $\sqrt{2}$.

We get

$$\sqrt{2} = 1+\frac{2}{5\,-}\,\frac{1}{6\,-}\,\frac{1}{6\,-}\,\cdots.$$

On using the cube of the initial matrix, we get

$$\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix} = \begin{pmatrix}7 & 10\\ 5 & 7\end{pmatrix},$$

$$\sqrt{2} = 1+\frac{5}{12\,+}\,\frac{1}{14\,+}\,\frac{1}{14\,+}\,\cdots.$$

4. The reduction of quadratic surds to non-periodic continued fractions by means of second-order matrices with variable elements

The results of the previous section enable us to obtain as many expansions as desired of irrational square roots as non-periodic continued fractions.

EXAMPLE 21. The matrix

$$\begin{pmatrix}1 & 2(n+1)\\ n+1 & 1\end{pmatrix}$$

satisfies conditions (5.63). In addition, it leads to the equation

$$\frac{P_n}{Q_n} = \frac{P_{n-1}+2(n+1)Q_{n-1}}{(n+1)P_{n-1}+Q_{n-1}},$$

from which it follows that $\lim_{n\to\infty} P_n/Q_n = \sqrt{2}$. The equivalent continued fraction (5.65) has the form

$$\sqrt{2} = 1+\frac{2}{3\,+}\,\frac{21}{5\,+}\,\frac{8.17}{7\,+}\,\frac{15.31}{9\,+}\,\frac{24.49}{11\,+}\,\cdots$$

$$\cdots+\frac{(n^2-1)(2n^2-1)}{2n+1}+\cdots.$$

Similarly, by starting from the matrix

$$\begin{pmatrix}n & 2(n+1)\\ n+1 & n\end{pmatrix},$$

we get the continued fraction

$$\sqrt{2} = \frac{4}{1\,+}\,\frac{21}{7\,+}\,\cdots+\frac{(n^2-1)(n^2+2n-1)}{2n^2-1}+\cdots.$$

5. Extraction of the root of any rational power by means of matrices

Extraction of the cubic root with the aid of matrices. Let us consider the three sequences $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, the terms of which are connected by the relationships:

$$\left.\begin{aligned} P_n &= aP_{n-1}+tQ_{n-1}+tR_{n-1}, \\ Q_n &= P_{n-1}+aQ_{n-1}+tR_{n-1}, \\ R_n &= P_{n-1}+Q_{n-1}+aR_{n-1} \end{aligned}\right\} \quad (n = 1, 2, \ldots). \tag{5.66}$$

These relationships may be written with the aid of matrices as

$$\begin{pmatrix} P_n \\ Q_n \\ R_n \end{pmatrix} = \begin{pmatrix} a & t & t \\ 1 & a & t \\ 1 & 1 & a \end{pmatrix} \begin{pmatrix} P_{n-1} \\ Q_{n-1} \\ R_{n-1} \end{pmatrix} \tag{5.67}$$

Let $x = \lim\limits_{n\to\infty} P_n/Q_n$ and $y = \lim\limits_{n\to\infty} Q_n/R_n$ exist and be finite. Now,

$$x = \sqrt[3]{t^2}, \quad y = \sqrt[3]{t}.$$

EXAMPLE 22. Let $a = 1$, $t = 2$. Then

$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$	1	5	19	73	281	1081
	1	4	15	58	223	858
	1	3	12	46	177	681
$\sqrt[3]{4}$ approximately	1	1·67	1·583	1·5870	1·58757	1·58737
$\sqrt[3]{2}$ approximately	1	1·33	1·250	1·2609	1·25989	1·259912

It is well known that $\sqrt[3]{2} \approx 1{\cdot}2599210$, $\sqrt[3]{4} \approx 1{\cdot}5874011$.

By utilizing the square, cube or a higher power of the initial matrix, the convergence of the process can be strengthened as desired.

Extraction of the fourth root with the aid of matrices. We generalize relationships (5.66) and consider the following equations:

$$\left.\begin{aligned} P_n &= aP_{n-1}+tQ_{n-1}+tR_{n-1}+tS_{n-1}, \\ Q_n &= P_{n-1}+aQ_{n-1}+tR_{n-1}+tS_{n-1}, \\ R_n &= P_{n-1}+Q_{n-1}+aR_{n-1}+tS_{n-1}, \\ S_n &= P_{n-1}+Q_{n-1}+R_{n-1}+aS_{n-1}. \end{aligned}\right| \quad (n = 1, 2, \ldots).$$

We can write these with the aid of matrices as:

$$\begin{pmatrix} P_n \\ Q_n \\ R_n \\ S_n \end{pmatrix} = \begin{pmatrix} a & t & t & t \\ 1 & a & t & t \\ 1 & 1 & a & t \\ 1 & 1 & 1 & a \end{pmatrix} \begin{pmatrix} P_{n-1} \\ Q_{n-1} \\ R_{n-1} \\ S_{n-1} \end{pmatrix}.$$

Now,

$$\lim_{n\to\infty} \frac{P_n}{S_n} = \sqrt[4]{t^3}, \quad \lim_{n\to\infty} \frac{Q_n}{S_n} = \sqrt[4]{t^2} = \sqrt{t}, \quad \lim_{n\to\infty} \frac{R_n}{S_n} \sqrt[4]{t},$$

if these limits exist and are finite. When $a = 1$, we have the matrix

$$\begin{pmatrix} 1 & t & t & t \\ 1 & 1 & t & t \\ 1 & 1 & 1 & t \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{matrix} 1 & 1+3t & 1+12t+3t^2 & 1+31t+31t^2+t^3 \\ 1 & 2+2t & 3+12t+t^2 & 4+40t+20t^2 \\ 1 & 3+t & 6+10t & 10+44t+10t^2 \\ 1 & 4 & 10+6t & 20+40t+4t^2. \end{matrix}$$

Hence we can obtain in particular the inequalities

$$\frac{t^2+12t+3}{6t+10} \leqq \sqrt{t} \leqq \frac{6t^2+10t}{t^2+12t+3} \quad (1 \leqq t \leqq 9),$$

$$\frac{6t^2+10t}{t^2+12t+3} \leqq \sqrt{t} \leqq \frac{t^2+12t+3}{6t+10} \quad (t \geqq 9).$$

The domain in which we can apply these approximations is clear from the following table:

t	$\frac{t^2+12t+3}{6t+10}$	$\sqrt{t}$	$\frac{6t^2+10t}{t^2+12t+3}$
1	1·000	1·000	1·000
2	1·409	1·414	1·419
3	1·714	1·732	1·750
4	1·971	2·000	2·030
5	2·200	2·236	2·273
6	2·413	2·449	2·486
7	2·615	2·646	2·676
8	2·810	2·828	2·847
9	3·000	3·000	3·000
10	3·186	3·162	3·139
11	3·368	3·317	3·266
12	3·549	3·464	3·381
13	3·727	3·606	3·488

Extraction of any rational root with the aid of matrices. It may be seen, on generalizing the method above described, that the nth-order square matrix

$$\begin{pmatrix} a & t & t \dots t \\ 1 & a & t \dots t \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 1 & 1 & 1 \dots a \end{pmatrix}$$

enables approximate values to be obtained for

$$\sqrt[n]{t}\,,\quad \sqrt[n]{t^2}\,,\ \dots,\ \sqrt[n]{t^{n-1}}\,.$$

For example, to evaluate $\sqrt[7]{4}$, we only need to carry out the following working:

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{matrix} 1 & 13 & 127 \\ 1 & 12 & 115 \\ 1 & 11 & 104 \\ 1 & 10 & 94 \\ 1 & 9 & 85 \\ 1 & 8 & 77 \\ 1 & 7 & 70 \end{matrix}$$

In particular, $\sqrt[7]{4} \approx 104/85 = 1{\cdot}223$ or $\sqrt[7]{4} \approx 85/70 \approx 1{\cdot}214$. The accurate value of this root is 1·219. . . .

The convergence of Euler's algorithm. The method of extraction of roots of any rational degree with the aid of matrices was first proposed by Euler (though not in matrix form). The conditions for its convergence were recently obtained by L. D. Eskin. In particular, he showed that Euler's algorithm is convergent for any $t > 0$ provided only that the zero approximation be chosen by a reliable method. Furthermore, Euler's algorithm is convergent for complex t, which enables it to be used for approximate extraction of roots of complex numbers.

6. Solution of cubic equations by means of matrices

The matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{5.68}$$

eads to the equations

$$x = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}} \quad \text{and} \quad y = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}. \tag{5.69}$$

On eliminating x from these equations, we arrive at a third degree equation in y. Consequently, the matrix (5.68) can serve, when the corresponding process is convergent, for approximate evaluation of one of the roots of a third degree equation.

We can readily obtain from (5.69) the equations

$$y = 1 + \frac{(a_{21} - a_{31})x + (a_{22} - a_{32})y + a_{23} - a_{33}}{a_{31}x + a_{32}y + a_{33}},$$

$$x = y + \frac{(a_{11} - a_{21})x + (a_{12} - a_{22})y + a_{13} - a_{23}}{a_{31}x + a_{32}y + a_{33}}.$$

Let us require that the condition $y^2 = x$ be satisfied. For this, it is sufficient that the equations obtained take the form

$$y = 1 + \frac{a_{23} - a_{33}}{a_{31}x + a_{32}y + a_{33}}, \quad x = y + \frac{(a_{23} - a_{33})y}{a_{31}x + a_{32}y + a_{33}}.$$

Now,

$$a_{21} = a_{31}, \quad a_{22} = a_{32}, \quad a_{11} = a_{21}, \quad a_{13} = a_{23},$$
$$a_{12} - a_{22} = a_{23} - a_{33}.$$

Matrix (5.68) takes the form

$$\begin{pmatrix} a_{11} & a_{22} + a_{13} - a_{33} & a_{13} \\ a_{11} & a_{22} & a_{13} \\ a_{11} & a_{22} & a_{33} \end{pmatrix}. \tag{5.70}$$

It leads to the equation

$$a_{11}y^3 + (a_{22} - a_{11})y^2 + (a_{33} - a_{22})y - a_{13} = 0. \tag{5.71}$$

Let us put, in particular, $a_{11} = a_{22} = 1$ and let us write $a_{33}-1 = p$, $a_{13} = -q$. The matrix (5.70) takes the form

$$\begin{pmatrix} 1 & -p-q & -q \\ 1 & 1 & -q \\ 1 & 1 & p+1 \end{pmatrix} . \tag{5.72}$$

It leads to the equation

$$y^3+py+q = 0 .$$

When $p = 0$, $q = -t$, the matrix (5.72) becomes the matrix for extracting the cube root.

Example 23. Let us apply matrix (5.72) for obtaining approximately one of the roots of the equation $x^3-x-1 = 0$. We have

$$\begin{array}{cccccccc} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} & \begin{array}{c} 4 \\ 3 \\ 2 \end{array} & \begin{array}{c} 12 \\ 9 \\ 7 \end{array} & \begin{array}{c} 37 \\ 28 \\ 21 \end{array} & \begin{array}{c} 114 \\ 86 \\ 65 \end{array} & \begin{array}{c} 351 \\ 265 \\ 200 \end{array} & \begin{array}{c} 1081 \\ 816 \\ 616 \end{array} \\ & 1 & 1{\cdot}5 & 1{\cdot}285 & 1{\cdot}333 & 1{\cdot}292 & 1{\cdot}325 & 1{\cdot}324 \end{array}$$

Notice that the convergence of this method is only rapid when the equation does not possess two roots close to one another in absolute value.

7. Recurrent series. The Bernoulli–Euler method

A power series $\sum\limits_{n=0}^{\infty} a_n x^n$, the coefficients of which, as from a certain n, are connected by the same linear relationship, is described as *recurrent*. The following theorem holds: *the necessary and sufficient condition for the sum of the series* $\sum\limits_{n=0}^{\infty} a_n x^n$, *convergent for* $|x| < 1$, *to be a rational function of* x *is that the series be recurrent.*

D. Bernoulli and Euler applied recurrent series to the solution of algebraic equations. Cauchy provided the basis of the method by indicating that, if x_1 is the root of least absolute value of the polynomial $Q(x)$, the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n x^n = \frac{P(x)}{Q(x)} ,$$

where $P(x)$ is a polynomial having no roots in common with $Q(x)$, and the degree of which does not exceed the degree of $Q(x)$, is given in accordance with d'Alembert's test by the relationship

$$\lim_{n\to\infty} \frac{a_{n+1}x_1}{a_n} = 1.$$

from which it follows that

$$x_1 = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}.$$

For instance, Euler solved the equation $x^3-3x+1 = 0$ by using the following expansion:

$$\frac{1}{1-3x+x^3} = 1+3x+9x^2+26x^3+75x^4+216x^5+$$
$$+622x^6+1791x^7+5157x^8+14849x^9+$$
$$+42756x^{10}+\ldots \equiv \sum_{n=0}^{\infty} a_n x^n.$$

Here $x_1 = \lim_{n\to\infty} a_n/a_{n+1}$, where x_1 is the root of least absolute value of the equation $x^3-3x+1 = 0$. The convergence of the process is clear from the following table:

n	$\frac{a_n}{a_{n+1}}$
1	0·333
2	0·333
3	0·346
4	0·3467
5	0·34722
6	0·34727
7	0·347292

On replacing x by $1/x$, the same method can be used to find approximately the root of greatest absolute value of the equation $Q(x) = 0$.

The convergence of the Bernoulli–Euler method is also only rapid when the equation does not have two roots close to one another in absolute value.

8. Connection between the Bernoulli–Euler method and matrix methods

Let us find by the Bernoulli–Euler method the root of least absolute value of the equation $x^3+px+q = 0$. For this, it is sufficient to expand $1/(q+px+x^3)$ as a recurrent series. We have

$$\frac{1}{q+px+x^3} = A_0+A_1x+A_2x^2+\ldots .$$

On equating coefficients of powers of x, we get

$$A_0 = \frac{1}{q}, \quad qA_1+pA_0=0, \quad qA_2+pA_1 = 0. \quad qA_3+pA_3+A_0 = 0.$$

and in the general case:

$$qA_n+pA_{n-1}+A_{n-3} = 0 \qquad (n \geqq 3).$$

This relationship can be written as the matrix equation:

$$\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q \\ -1 & 0 & -p \end{pmatrix} \begin{pmatrix} A_{n-3} \\ A_{n-2} \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} qA_{n-2} \\ qA_{n-1} \\ qA_n \end{pmatrix} .$$

But this equation gives only one new coefficient A_n, while the two preceding ones are repeated. On taking the cube of the matrix composed of the coefficients, we at once obtain three new coefficients:

$$\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q \\ -1 & 0 & -p \end{pmatrix}^3 \begin{pmatrix} A_{n-3} \\ A_{n-2} \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} q^3A_n \\ q^3A_{n+1} \\ q^3A_{n+2} \end{pmatrix},$$

i.e.

$$\begin{pmatrix} q^2 & 0 & pq^2 \\ -pq & q^2 & -p^2q \\ p^2 & -pq & p^3+q^2 \end{pmatrix} \begin{pmatrix} A_{n-3} \\ A_{n-2} \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} q^3A_n \\ q^3A_{n+1} \\ q_3A_{n+2} \end{pmatrix} .$$

On applying this method to the equation $x^3-3x-1 = 0$, we get

$$\begin{pmatrix} -1 & 0 & 3 \\ -3 & -1 & 9 \\ -9 & -3 & 26 \end{pmatrix} \begin{matrix} 1 & 26 & 622 & 14849 \\ 3 & 75 & 1791 & 42756 \\ 9 & 216 & 5157 & 123111 \end{matrix} .$$

An even higher power of the initial matrix can be taken; this still further strengthens the convergence, though not all the coefficients of the recurrent series are utilized in this case. The elements of the matrices obtained can be divided by the same number, since we only need the ratios of neighbouring coefficients. But the series obtained here will in general differ from the recurrent series, into which $1/(q+px+x^3)$ is expanded.

On replacing x by $1/x$, the same method can be used to evaluate the root of greatest absolute value of the equation $x^3+px+q = 0$ (if the corresponding process is convergent).

9. Solution of higher degree equations by means of matrices

We can use a method similar to the above to form nth-order matrices that can be employed for the approximate evaluation of the roots of an nth-degree equation. For example, the matrix

$$\begin{pmatrix}
k & la_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & k & la_n & \dots & 0 & 0 & 0 & 0 & 0 \\
\hdotsfor{9} \\
0 & 0 & 0 & \dots & la_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & k & la_n & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & k & la_n & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & k & 0 & la_n \\
-la_0 & -la_1 & -la_2 & \dots & -la_{n-5} & -la_{n-4} & -la_{n-3} & k-la_{n-1} & -la_{n-2} \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & la_n & k
\end{pmatrix}$$

can serve, when the corresponding process is convergent, for approximate evaluation of one of the roots of the equation

$$a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0 = 0.$$

For example, in the case of the equation

$$x^4-8x^3+x^2-x+1 = 0,$$

we obtain on putting $k = l = 1$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 9 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{array}{rrrrrrrr} 1 & (2) & 1 & 2 & 4 & 11 & 58 & 455 \\ 1 & (2) & 1 & 2 & 7 & 47 & 397 & 3500 \\ 1 & (8) & 4 & 35 & 310 & 2753 & 24463 & 217403 \\ 1 & (2) & 1 & 5 & 40 & 350 & 3103 & 27566 \\ 1 & & 4 & 7 & 7{\cdot}75 & 7{\cdot}866 & 7{\cdot}8837 & 7{\cdot}8866 \end{array}$$

The exact value of this root is 7·8873. . . .

10. The idea behind Jacobi's algorithm

Euler posed and partially solved the problem of finding the solution in integers of the equation $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$. Jacobi posed the analogous problem for the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$$

and developed for this purpose a special algorithm, which is a generalization of the algorithm of continued fractions. In particular, Jacobi assigned to this algorithm the following form:

$$\left.\begin{aligned} u_{n+1} &= v_n - p_n u_n, \\ v_{n+1} &= w_n - q_n u_n, \\ w_{n+1} &= u_n. \end{aligned}\right\} \qquad (5.73)$$

Here, p_n and q_n are positive integers, chosen in such a way that u_{n+1} and v_{n+1} are the least possible positive numbers. With the aid of the numbers p_n and q_n, we form the following relationships:

$$\left.\begin{aligned} A_n &= q_n A_{n-1} + p_n A_{n-2} + A_{n-3}, \\ B_n &= q_n B_{n-1} + p_n B_{n-2} + B_{n-3}, \\ C_n &= q_n C_{n-1} + p_n C_{n-2} + C_{n-3}, \end{aligned}\right\} \qquad (5.74)$$

We suppose that

$$\begin{pmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, when Jacobi's algorithm is convergent,

$$\lim_{n\to\infty} \frac{A_n}{C_n} = \frac{u_1}{w_1}, \qquad \lim_{n\to\infty} \frac{B_n}{C_n} = \frac{v_1}{w_1}.$$

Jacobi's algorithm enables us to find fairly small integers A_n, B_n, C_n, approximately proportional to the given large numbers u_1, v_1, w_1.

EXAMPLE 24. Let us solve approximately the equation

$$x : y : z \approx 49 : 59 : 75.$$

We have, on using Jacobi's algorithm,

n	u_n	v_n	w_n	p_n	q_n	A_n	B_n	C_n
-2						1	0	0
-1						0	1	0
0						0	0	1
1	49	59	75	1	1	1	1	1
2	10	26	49	2	4	4	5	6
3	6	9	10	1	1	5	6	8
4	3	4	6	1	2	15	18	23
5	1	0	3	0	3	49	59	75
6	0	0	1					

Thus, for example, $5 : 6 : 8 \approx 15 : 18 : 23 \approx 49 : 59 : 75$.

The order of filling up this table is as follows: we first evaluate $u_1 = 49$, $v_1 = 59$, $w_1 = 75$ and A_n, B_n, C_n, with $n = -2, -1, 0$. Next, in accordance with algorithm (5.73), we evaluate p_1, q_1, u_2, v_2, w_2. Then, with the aid of relations (5.74), we evaluate A_1, B_1, C_1. Next, in accordance with algorithm (5.73), we evaluate p_2, q_2, u_3, v_3, w_3. With the aid of relationship (5.74), we evaluate A_2, B_2, C_2, and so on.

CHAPTER VI

SOME SPECIAL CONSTANTS AND FUNCTIONS

THE present chapter contains information on real constants and some real functions commonly encountered in analysis.†

Our treatment must take into account the fact that differing definitions are current in mathematical literature. For instance, the *generating function* of a sequence of functions is usually defined as the function of two variables $f(x, y)$ for which the relationship

$$f(x, y) = \sum_{n=1}^{\infty} \varphi_n(x) y^n,$$

is satisfied in some domain of the x, y plane, i.e. the $\varphi_n(x)$ are the *coefficients* of the expansion of $f(x, y)$ as a series in powers of y (cf. the definitions of the generating functions of Bessel functions and Legendre polynomials). In other cases, $f(x, y)$ is called the *generating function* of a sequence if

$$f(x, y) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)}{n!} y^n$$

(cf. the definitions of the generating functions of Bernoulli and Euler polynomials).

There is also a lack of uniformity as regards the notation for various functions. In such cases the different notations will be quoted, though only one of them will be utilized.

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† Information concerning functions of a complex variable will be given in a later volume.

§ 1. Various constants and expressions

1. Some well-known constants

1°. The number π: $\pi = 3{\cdot}14159\ 26535\ 89793\ldots$

The number π made its appearance in connection with the calculation of the length of a circumference by finding successively the perimeters of the inscribed and circumscribed regular polygons with 2^n sides. π is a *transcendental* number.

Some approximations for the value of π:

$\pi \approx 22/7$ (*Archimedes' number*);

$\pi \approx 355/113$ (*Metsiev's number*).

Some representations of π as series and products. On putting $x = 1$ in the power series expansion of arc tan x, we get the *Leibniz series*

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+(-1)^k\frac{1}{2k+1}+\ldots. \tag{6.1}$$

With $x = 1/\sqrt{3}$, we get from the same expansion

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1\frac{1}{3}\frac{1}{3}+\frac{1}{5}\frac{1}{3^2}-\frac{1}{7}\frac{1}{3^3}+\ldots\right). \tag{6.2}$$

Since $\pi = 16 \operatorname{arc\,tan}(1/5) - 4 \operatorname{arc\,tan}(1/239)$ (*Machin's formula*), we have

$$\pi = 16\left(\frac{1}{5}-\frac{1}{3}\frac{1}{5^3}+\frac{1}{5}\frac{1}{5^5}-\frac{1}{7}\frac{1}{57}+\frac{1}{9}\frac{1}{59}-\ldots\right)- \\ -4\left(\frac{1}{239}-\frac{1}{3}\frac{1}{239^3}+\ldots\right). \tag{6.3}$$

Wallis's formula is

$$\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2\frac{1}{2n+1}. \tag{6.4}$$

As a continued fraction:

$$\frac{\pi}{4} = \frac{1}{1+}\,\frac{1^2}{3+}\,\frac{2^2}{5+}\ldots\frac{n^2}{+2n+1+}\ldots. \tag{6.5}$$

Other relationships are:

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \left[1 - \frac{1}{(2n+1)^2}\right]. \tag{6.6}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \arctan \frac{1}{2n^2}. \tag{6.7}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \arctan \frac{1}{n^2+n+1}. \tag{6.8}$$

$$\frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{1}{(4n+1)(4n+3)}. \tag{6.9}$$

$$\frac{\pi}{4} = \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{2\nu-1} + (-1)^n n \sum_{\mu=1}^{m} \frac{E_{2\mu-2}}{(2n)^{2\mu}} + \frac{\theta(-1)^n E_{2m}}{(2n)^{2m+2}}. \tag{6.10}$$

$$0 < \theta < 1.$$

$$\frac{\pi}{2} = \sum_{\nu=1}^{n} \frac{2n}{n^2+\nu^2} - \frac{2\pi}{e^{2\pi n}-1} + \frac{1}{2n} +$$

$$+ \sum_{\mu=1}^{m} \frac{(-1)^{\mu-1} B_{4\mu-2}}{(2n^2)^{2\mu-1}} \frac{1}{2\mu-1} + \frac{\theta(-1)^m}{(2n^2)^{m+1}} \frac{B_{4m+2}}{2m+1}. \tag{6.11}$$

$$\frac{\pi}{2} = \frac{2}{\sqrt{2}} \frac{2}{\sqrt{2+\sqrt{2}}} \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots. \tag{6.12}$$

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \cos \frac{\pi}{2^{n+1}}. \tag{6.13}$$

$$\frac{\pi^2}{4} = \frac{39}{16} + \frac{1}{1^2.2^2.3^2} + \frac{1}{2^2.3^2.4^2} + \frac{1}{3^2.4^2.5^2} + \cdots. \tag{6.14}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{6.15}$$

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}. \tag{6.16}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}. \tag{6.17}$$

$$\frac{\pi^2}{12} = \frac{1}{2}(\ln 2)^2 + \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}. \tag{6.18}$$

$$\frac{3\pi^2}{32} = 1-2\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^3}. \tag{6.19}$$

$$\frac{\pi^2}{16} = \frac{1}{2}+\frac{1}{1^2 3^2}+\frac{1}{3^2 5^2}+\frac{1}{5^2 7^2}+\ldots. \tag{6.20}$$

$$\frac{3\pi^2}{256} = \frac{1}{9}+\frac{1}{1^2 3^2 5^2}+\frac{1}{3^2 5^2 7^2}+\frac{1}{5^2 7^2 9^2}+\ldots. \tag{6.21}$$

Inequalities connected with the number π. If $a_n \geqq 0$ $(n=1, 2, \ldots)$, but not all are zero, we have

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 \leqq \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2, \tag{6.22}$$

π^2 being the best constant in the sense that there exists a sequence a_n for which equality is attained. If $f(x) \geqq 0$, $f(x) \not\equiv 0$, $f \in L^2(0, \infty)$, $xf \in L_2(0, \infty)$, we have

$$\left\{\int_0^{\infty} f(x)dx\right\}^4 \leqq \pi^2 \left\{\int_0^{\infty} f^2(x)\, dx\right\}\left\{\int_0^{\infty} x^2 f^2(x)dx\right\}, \tag{6.23}$$

where π^2 is the best constant in the sense that there exist $f(x)$ for which equality is attained.

Integral forms of π:

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\tan x}{x}\, dx. \tag{6.24}$$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin x \cos mx}{x}\, dx,\ m < 1. \tag{6.25}$$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx. \tag{6.26}$$

$$\frac{1}{2}\sqrt{\frac{\pi}{2}} = \int_0^{\infty} \sin(x^2)\, dx = \int_0^{\infty} \cos(x^2)\, dx. \tag{6.27}$$

$$\sqrt{\frac{\pi}{2}} = \int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx. \tag{6.28}$$

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2}dx. \tag{6.29}$$

$$\frac{\sqrt{\pi}}{4} = \int_0^\infty x^2e^{-x^2}dx. \tag{6.30}$$

$$\frac{\pi^2}{6} = \int_0^\infty \frac{x\,dx}{e^x-1}. \tag{6.31}$$

$$\frac{\pi^2}{12} = \int_0^\infty \frac{x\,dx}{e^x+1}. \tag{6.32}$$

$$\frac{\pi^2}{6} = \int_0^1 \frac{\ln x\,dx}{x-1}. \tag{6.33}$$

$$\frac{\pi^2}{8} = \int_0^1 \frac{\ln x\,dx}{x^2-1}. \tag{6.34}$$

$$\frac{\pi^2}{12} = \int_0^1 \frac{\ln(1+x)\,dx}{x}. \tag{6.35}$$

2°. The number *e:*

$$e = 2{\cdot}7182818284\,59045\ldots$$

Euler's number e is defined as the limit

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}. \tag{6.36}$$

e is a *transcendental* number.

Expressions for e, e^2, e^{-1}, $(1+e)^2$ as series and products:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{6.37}$$

$$\left.\begin{aligned} e &= \frac{1}{p_m} \sum_{n=0}^{\infty} \frac{n^m}{n!}. \\ p_{m+1} &= 1 + m p_1 + \frac{m+(m-1)}{1.2} p_2 + \dots + p_m; \\ p_1 = 1,\ & p_2 = 2,\ p_3 = 5,\ p_4 = 15,\ p_5 = 52, \dots . \end{aligned}\right\} \tag{6.38}$$

$$e = \frac{2}{1}\left(\frac{4}{3}\right)^{\frac{1}{2}} \left(\frac{6.8}{5.7}\right)^{\frac{1}{4}} \left(\frac{10.12.14.16}{9.11.13.15}\right)^{\frac{1}{8}} \dots . \tag{6.39}$$

$$e^2 = \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n(n+1)}{n!}. \tag{6.40}$$

$$(1+e)^2 = \sum_{n=0}^{\infty} \frac{2+2^n}{n!}. \tag{6.41}$$

$$e^{\frac{1}{2}} = \lim_{n\to\infty} \sqrt[n+1]{\binom{n}{0}\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}. \tag{6.42}$$

$$e^{-1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}. \tag{6.43}$$

$$e^{-1} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}. \tag{6.44}$$

$$2e^{-1} = \lim_{n\to\infty} \frac{\sqrt[n]{a(a+d)(a+2d)\dots[a+(n-1)d]}}{\frac{1}{n}\sum_{k=0}^{n-1}(a+kd)} \tag{6.45}$$

$$e = \frac{1}{1-}\,\frac{2}{3+}\,\frac{1}{6+}\,\frac{1}{10+}\dots+\frac{1}{2(2n+1)+}\dots \tag{6.46}$$

(continued fraction).

Some inequalities. When $0 \leqq t \leqq n$:

$$1 - \frac{t}{n} \leqq e^{-\frac{t}{n}}. \tag{6.47}$$

$$1 + \frac{t}{n} \leqq e^{\frac{t}{n}}. \tag{6.48}$$

When $u_n > 0$:

$$0 \leqq 1 - e^t\left(1-\frac{t}{n}\right)^n \leqq \frac{t^2}{n}. \tag{6.49}$$

$$\frac{\sum\limits_{n=1}^{\infty} \sqrt[n]{u_1 u_2 \ldots u_n}}{\sum\limits_{n=1}^{\infty} u_n} \leqq e. \tag{6.50}$$

$$\lim_{n\to\infty} \sup \frac{u_1 + u_{n+1}}{u_n} \geqq e. \tag{6.51}$$

Minima:

$$\min_{0 \leqq x < \infty} x^x = e^{-1}. \tag{6.52}$$

$$\min_{0<x<\infty} \frac{x}{\ln x} = e. \tag{6.53}$$

The number e is the base of logarithms with which the inequality is satisfied:

$$\log ax \leqq x - 1 \quad \text{(Ostrogradskii)}. \tag{6.54}$$

The functions e^x and e^{-x} are defined as the limits

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n, \quad e^{-x} = \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n. \tag{6.55}$$

3°. The Euler–Mascheroni constant C:

$$C = 0{\cdot}5772156649\ 015325\ldots$$

The *Euler–Mascheroni constant* C is defined as the limit

$$C = \lim_{m\to\infty}\left(\sum_{n=1}^{m}\frac{1}{n} - \ln m\right), \tag{6.56}$$

the existence of which follows from the convergence of the series with general term

$$g(n) = \frac{1}{n} - \int_n^{n+1} t^{-1}\,dt.$$

It is unknown whether C is rational or irrational.

Integral forms. Since

$$1+\frac{1}{2}+\,\ldots\,+\frac{1}{m}=\int_0^1\frac{1-t^m}{1-t}\,dt=\int_0^m\left[1-\left(1-\frac{y}{m}\right)^m\right]y^{-1}\,dy.$$

and

$$-\ln m = -\int_1^m y^{-1}\,dy$$

we have

$$\sum_{n=1}^{m}\frac{1}{n}-\ln m=\int_0^1\left[1-\left(1-\frac{y}{m}\right)^m\right]y^{-1}dy-\int_1^m\left(1-\frac{y}{m}\right)^m y^{-1}dy$$

from which we find, on passing to the limit, that

$$C=\int_0^1(1-e^{-y})y^{-1}\,dy-\int_1^\infty e^{-y}y^{-1}\,dy \qquad (6.57)$$

or

$$C=\int_0^1\left(1-e^{-t}-e^{-\frac{1}{t}}\right)t^{-1}\,dt. \qquad (6.58)$$

It follows from (6.58) that

$$C=\int_0^\infty[(1-e^{-t})^{-1}-t^{-1}]\,e^{-t}\,dt. \qquad (6.59)$$

Other integral forms.

$$C=-\int_0^\infty e^{-t}\ln t\,dt. \qquad (6.60)$$

$$C=\int_0^1\left[\frac{1}{\ln t}+\frac{1}{1-t}\right]dt. \qquad (6.61)$$

$$C=-\int_0^\infty\left(\cos t-\frac{1}{1+t}\right)\frac{dt}{t}. \qquad (6.62)$$

$$C=1-\int_0^\infty\left(\frac{\sin t}{t}-\frac{1}{1+t}\right)\frac{dt}{t}. \qquad (6.63)$$

$$C=\int_0^\infty\left(\frac{1}{1+t}-e^{-t}\right)\frac{dt}{t}. \qquad (6.64)$$

$$C=\int_{0}^{\infty}\left(\frac{1}{1+t^{2}}-e^{-t}\right)\frac{dt}{t}. \tag{6.65}$$

$$C=\int_{0}^{\infty}\left(\frac{1}{e^{t}-1}-\frac{\cos t}{t}\right)dt. \tag{6.66}$$

$$C=-\int_{0}^{1}\ln\ln\frac{1}{t}\,dt. \tag{6.67}$$

$$C=\int_{0}^{\infty}\left(\frac{2}{\pi}\operatorname{arc\,cot} t-e^{-pt}\right)\frac{dt}{t}-\ln p. \tag{6.68}$$

$$C=\int_{0}^{\infty}\left(\frac{2}{\pi}\operatorname{arc\,cot} t-\cos pt\right)dt-\ln p. \tag{6.69}$$

$$C=-\frac{2}{\pi}\int_{0}^{\infty}\sin t\ln t\,\frac{dt}{t}. \tag{6.70}$$

The asymptotic expansion is

$$C=\sum_{n=1}^{m-1}\frac{1}{n}-\ln m+\frac{1}{2m}+\frac{1}{12m^{2}}-\frac{1}{120m^{4}}+\dots$$

$$\dots+\frac{B_{2n}}{2n}\frac{1}{m^{2n}}+\theta\,\frac{B_{2n+2}}{2n+2}\frac{1}{m^{2n+2}},\quad 0<\theta<1. \tag{6.71}$$

Infinite products expressible in terms of e and C are:

$$\prod_{1}^{\infty}\frac{e^{\frac{1}{n}}}{1+\frac{1}{n}}=\lim_{m\to\infty}\frac{e^{\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m}-\ln m\right)+\ln m}}{\prod_{1}^{m}\left(1+\frac{1}{n}\right)}=$$

$$e^{C}\lim_{m\to\infty}\frac{m}{\prod_{1}^{\infty}\left(1+\frac{1}{n}\right)}=e^{C}. \tag{6.72}$$

$$C=\lim_{t\to 1-0}\left[(1-t)\left(\frac{t}{1-t}+\frac{t^{2}}{1-t^{2}}+\dots+\frac{t^{n}}{1-t^{n}}-\ln\frac{1}{1-t}\right)\right]. \tag{6.73}$$

4°. Catalan's constant G:

$$G=0{\cdot}91596\,55941\,77219\,0\dots.$$

Catalan's constant G is defined as the number

$$G = \int_0^1 \frac{\arc \tan x}{x}\, dx. \tag{6.74}$$

An expression for Catalan's constant as a series can be obtained by expanding x^{-1} arc tan x in (6.74) as a power series:

$$G = \sum_{m=0}^{\infty} \frac{-1m}{(2m+1)^2}. \tag{6.75}$$

Other integral forms are:

$$G = \frac{1}{2}\int_0^{\pi/2} \frac{t\, dt}{\sin t}. \tag{6.76}$$

$$G = \frac{\pi}{4}\ln 2 + 2\int_0^{\pi/4} t \tan t\, dt. \tag{6.77}$$

$$G = -\frac{\pi}{4}\ln 2 + 2\int_0^{\pi/4} t \cot t\, dt. \tag{6.78}$$

$$\pm G = -\frac{\pi}{4}\ln 2 + \int_0^{\pi/2} \frac{t\, dt}{(\cos t + \sin t)\sin t}. \tag{6.79}$$

$$G = \frac{\pi}{8}\ln 2 + \int_0^{\pi/4} \frac{t\, dt}{(\sin t + \cos t)\sin t}. \tag{6.80}$$

$$G = -\frac{\pi}{2}\ln 2 - 2\int_0^{\pi/4} \ln \sin t\, dt. \tag{6.81}$$

$$G = -\frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln \cos t\, dt. \tag{6.82}$$

$$\pm G = \frac{1}{2}\int_0^{\pi/2} \ln (1 \pm \cos t)\, dt + \frac{\pi}{4}\ln 2. \tag{6.83}$$

$$G = \int_0^{\pi/2} \ln (1 + \tan t)\, dt - \frac{\pi}{4}\ln 2. \tag{6.84}$$

$$G = -\int_0^1 \frac{\ln t}{1+t^2}\, dt. \tag{6.85}$$

$$G = -\frac{\pi}{2}\ln 2 - 2\int_0^{\sqrt{\frac{1}{2}}} \frac{\ln t\, dt}{\sqrt{1-t^2}}. \tag{6.86}$$

$$G = \int_0^\infty \ln(1+t)\frac{dt}{1+t^2} - \frac{\pi}{4}\ln 2. \tag{6.87}$$

$$G = -\frac{\pi}{4}\ln 2 - \frac{1}{2}\int_0^1 \frac{\ln(1+t)}{\sqrt{1-t^2}}\,dt. \tag{6.88}$$

$$G = \frac{\pi}{8}\ln 2 - \int_0^1 \frac{\ln(1-t)}{1+t^2}\,dt. \tag{6.89}$$

$$G = \int_0^1 \ln\frac{1+t}{1-t}\,\frac{dt}{1+t^2}. \tag{6.90}$$

$$G = \frac{3\pi}{8}\ln 2 - \int_0^1 \ln\frac{1+t^2}{1+t}\,\frac{dt}{1+t^2}. \tag{6.91}$$

$$G = \pm\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^1 \frac{\operatorname{arc\,cos} t}{1\pm t}\,dt. \tag{6.92}$$

$$G = \int_0^\infty \frac{\operatorname{arc\,cot} t}{1\pm t}\,dt \mp \frac{\pi}{4}\ln 2. \tag{6.93}$$

$$G = \int_0^1 \frac{\operatorname{arc\,tan} t}{t(1+t)}\,dt + \frac{\pi}{8}\ln 2. \tag{6.94}$$

$$G = -\int_0^\infty \frac{\operatorname{arc\,tan} t}{1-t^2}\,dt. \tag{6.95}$$

$$G = \frac{\pi}{8}\ln 2 + \int_0^1 \left(\frac{\pi}{4} - \operatorname{arc\,tan} t\right)\frac{dt}{1-t}. \tag{6.96}$$

$$G = 2\int_0^1 \left(\frac{\pi}{4} - \operatorname{arc\,tan} t\right)\frac{1+t}{1-t}\,\frac{dt}{1+t^2} - \frac{\pi}{4}\ln 2. \tag{6.97}$$

$$G = \frac{\pi^2}{16} + \frac{1}{4}\int_0^\infty (\operatorname{arc\,tan} t)^2 \frac{dt}{t^2\sqrt{1+t^2}}. \tag{6.98}$$

$$G = \frac{\pi^2}{16} + \frac{1}{4}\int_0^\infty (\operatorname{arc\,cot} t)^2 \frac{t\,dt}{\sqrt{1+t^2}}. \tag{6.99}$$

Expressions in terms of complete elliptic integrals (see (6.380) and (6.381)):

$$G = \frac{1}{2}\int_0^1 K(k)\,dk. \tag{6.100}$$

$$G = \int_0^1 E(k)\,dk - 1. \tag{6.101}$$

2. Soumne merical expressions

1°. Factorials. When n is a positive integer:

$$n! = 1.2.3 \dots n. \tag{6.102}$$

By definition,

$$0! = 1.$$

The gamma function is a generalization of the factorial (see § 4, sec. 5).

When m is a positive integer:

$$(2m)!! = 2.4.6 \dots 2m. \tag{6.103}$$

$$(2m+1)!! = 1.3.5 \dots (2m+1). \tag{6.104}$$

$$(2m)! = 2^m m!(2m-1)!! \tag{6.105}$$

The number $n!!$ is called the *double factorial* of n. Expressions for factorials, inverse factorials and their sums:

$$\frac{1}{nn!} = \sum_{m=1}^{\infty} \frac{1}{m(m+1)\dots(m+n)}. \tag{6.106}$$

$$(n+1)! = 1 + \sum_{k=1}^{n} kk! \tag{6.107}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n)!} = \frac{e+e^{-1}}{2} = \cosh 1 = 1{\cdot}54308\dots. \tag{6.108}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n-1)!} = \frac{e-e^{-1}}{2} = \sinh 1 = 1{\cdot}175201\dots. \tag{6.109}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2(n)!} = \cos 1 = 0{\cdot}54030\dots. \tag{6.110}$$

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)!} = \sin 1 = 0{\cdot}84147\dots. \tag{6.111}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n!)^2} = I_0(2) = 2{\cdot}27958530\dots. \tag{6.112}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} = I_1(2) = 1{\cdot}590636855\dots. \tag{6.113}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!(n+k)!} = I_k(2). \tag{6.114}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} = J_0(2) = 0{\cdot}22389078\ldots. \tag{6.115}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} = J_1(2) = 0{\cdot}57672481\ldots. \tag{6.116}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} = J_k(2). \tag{6.117}$$

Inequalities and asymptotic formulae:

$$\sqrt{\frac{4}{5}}\, e\sqrt{n}\left(\frac{n}{e}\right)^n < n! < e\sqrt{n}\left(\frac{n}{e}\right)^n. \tag{6.118}$$

$$n! \sim \left(\frac{n}{e}\right)^n. \tag{6.119}$$

$$\sqrt{\frac{(2n)!}{(n!)2}} \sim 4. \tag{6.120}$$

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\ldots\right) \text{ (Stirling).} \tag{6.121}$$

$$(2n-1)!! = \sqrt{2}\,(2n)^n e^{-n+\frac{0_n}{12n}}, \quad |0_n| < 1. \tag{6.122}$$

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \frac{0}{e\;24n+12} \text{ (Gauss).} \tag{6.123}$$

A formula for the logarithm as a factorial:

$$\ln n! = \sum_{p \leqq n} \ln p\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots\right), \tag{6.124}$$

where the summation is over all primes p not exceeding n. ($[x]$ is the integral part of x. See § 3, sec. 1, 3°.)

Divisors of a factorial. The greatest index with which a prime p appears as a factor in $n!$ is

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\ldots+\left[\frac{n}{p^k}\right] \tag{6.125}$$

where $p^k \leqq n$, but $p^{k+1} > n$.

For example, the highest power of the number 2 by which 50! is divisible is

$$\frac{50}{2}+\left[\frac{50}{4}\right]+\left[\frac{50}{8}\right]+\left[\frac{50}{16}\right]+\left[\frac{50}{32}\right] = 47.$$

Thus 50! is divisible by 2^{47}.

2°. The Kronecker delta δ_n^m (or δ_{mn}) is defined by

$$\delta_n^m = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases} \tag{6.126}$$

The integral form is:

$$\delta_n^m = \int_a^n \varphi_m(x)\varphi_n(x)\,dx \tag{6.127}$$

where $\{\varphi_k(x)\}$ is a system of functions orthonormal in $[a, b]$.

For instance,

$$\delta_n^m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx\,dx. \tag{6.128}$$

3°. The binomial coefficients $\binom{n}{m}$ or C_n^m, where n is any real number and m is a positive integer, are defined by

$$\binom{n}{m} = \frac{n(n-1)\ldots(n-m+1)}{m!}. \tag{6.129}$$

When n is a positive integer:

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}. \tag{6.130}$$

The beta function is a generalization of the binomial coefficient (see § 4, sec. 5, 2°).

Some relationships between binomial coefficients (n, m integers):

$$\binom{n}{m} = 0 \quad \text{for} \quad n > 0, \quad m > n. \tag{6.131}$$

$$\binom{n}{m} = \binom{n}{n-m}. \tag{6.132}$$

$$\binom{n}{m}+\binom{n}{m+1} = \binom{n+1}{m+1}. \tag{6.133}$$

$$\binom{n}{0}+\binom{n+1}{1}+\ldots+\binom{n+k}{k}=\binom{n+1+k}{k}. \quad (6.134)$$

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^k\binom{n}{k}=(-1)^k\binom{n-1}{k}. \quad (6.135)$$

$$\binom{2n+1}{1}+\binom{2n+1}{3}+\ldots+\binom{2n+1}{2n+1}=2^{2n}. \quad (6.136)$$

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots=2^{n-1}. \quad (6.137)$$

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots=2^{n-1}. \quad (6.138)$$

$$\binom{2n}{0}+\binom{2n}{2}+\ldots+\binom{2n}{n-1}=2^{2n-2}\ (n \text{ is odd}). \quad (6.139)$$

$$\binom{2n}{1}+\binom{2n}{3}+\ldots+\binom{2n}{n-1}=2^{2n-2}\,(n \text{ is even}). \quad (6.140)$$

$$\binom{n}{1}-3\binom{n}{3}+3^2\binom{n}{5}-3^3\binom{n}{7}+\ldots=$$

$$=(3-1)^{n+1}\frac{2^n}{\sqrt{3}}\sin\frac{2n\pi}{3}. \quad (6.141)$$

$$\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right). \quad (6.142)$$

$$\binom{n}{1}+\binom{n}{4}+\binom{n}{7}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{(n+4)\pi}{3}\right). \quad (6.143)$$

$$\binom{n}{2}+\binom{n}{5}+\binom{n}{8}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{(n+2)\pi}{3}\right). \quad (6.144)$$

$$\binom{n}{0}+\binom{n}{4}+\binom{n}{8}+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right). \quad (6.145)$$

$$\binom{n}{1}+\binom{n}{5}+\binom{n}{9}+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right). \quad (6.146)$$

$$\binom{n}{2}+\binom{n}{6}+\binom{n}{10}+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right). \quad (6.147)$$

$$\binom{n}{3}+\binom{n}{7}+\binom{n}{11}+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right). \quad (6.148)$$

$$\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\binom{n}{6}+\ldots=2^{\frac{n}{2}}\cos\frac{n\pi}{4}. \quad (6.149)$$

$$\binom{n}{1}-\binom{n}{3}+\binom{n}{5}-\binom{n}{7}+\ldots=2^{\frac{n}{2}}\sin\frac{n\pi}{4}. \quad (6.150)$$

$$\binom{n}{0}+2\binom{n}{1}+3\binom{n}{2}+\ldots+(n+1)\binom{n}{n}=(n+2)2^{n-1}. \quad (6.151)$$

$$\binom{n}{1}-2\binom{n}{2}+3\binom{n}{3}-\ldots+(-1)^{n-1}n\binom{n}{n}=0\ (n\neq 1). \quad (6.152)$$

$$\frac{1}{2}\binom{n}{1}-\frac{1}{3}\binom{n}{2}+\ldots+\frac{(-1)^{n+1}}{n+1}\binom{n}{n}=\frac{n}{n+1}. \quad (6.153)$$

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\ldots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}. \quad (6.154)$$

$$2\binom{n}{0}+\frac{2^2}{2}\binom{n}{1}+\frac{2^3}{3}\binom{n}{2}+\ldots+\frac{2^{n+1}}{n+1}\binom{n}{n}=\frac{3^{n+1}-1}{n+1}. \quad (6.155)$$

$$\binom{n}{1}-\frac{1}{2}\binom{n}{2}+\frac{1}{3}\binom{n}{3}-\ldots+\frac{(-1)^{n-1}}{n}\binom{n}{n}=$$

$$=1+\frac{1}{2}+\ldots+\frac{1}{n}. \quad (6.156)$$

$$\binom{n+1}{2}+2\left[\binom{n}{2}+\binom{n-1}{2}+\ldots+\binom{2}{2}\right]=$$

$$=1^2+2^2+\ldots+n^2. \quad (6.157)$$

$$\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\ldots+\binom{n}{k}\binom{m}{0}=\binom{n+m}{k}. \quad (6.158)$$

$$\binom{n}{0}\binom{n}{k}+\binom{n}{1}\binom{n}{k+1}+\ldots+\binom{n}{n-k}\binom{n}{n}=$$

$$=\binom{2n}{n-k}=\frac{(2n)!}{(n-k)!(n+k)!}. \quad (6.159)$$

$$\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}=\frac{(2n)!}{n(!)^2}. \quad (6.160)$$

$$\binom{2n}{0}^2-\binom{2n}{1}^2+\ldots+\binom{2n}{2n}^2=(-1)^n\binom{2n}{n}=$$

$$=(-1)^n\frac{(2n)!}{(n!)^2}. \quad (6.161)$$

$$\binom{2n+1}{0}^2-\binom{2n+1}{1}^2+\ldots-\binom{2n+1}{2n+1}^2=0. \quad (6.162)$$

$$\binom{n}{1}^2+2\binom{n}{2}^2+\ldots+n\binom{n}{n}^2=$$

$$=(2n-1)\binom{2n-2}{n-1}=\frac{(2n-1)!}{[(n-1)!]^2}. \quad (6.163)$$

Binomial and polynomial formulae:

$$(a_1+a_2)^n=\sum_{\nu_1+\nu_2=n}\frac{n!}{\nu_1!\nu_2!}a_1^{\nu_1}a_2^{\nu_2}. \quad (6.164)$$

$$(a_1+a_2=\ldots+a_p)^n=$$

$$=\sum_{\nu_1+\nu_2+\ldots+\nu_p=n}\frac{n!}{\nu_1!\nu_2!\ldots\nu_p!}a_1^{\nu_1}a_2^{\nu_2}\ldots a_p^{\nu_p}. \quad (6.165)$$

Some identities in connection with binomial coefficients:

$$\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k}=x. \quad (6.166)$$

$$\sum_{k=0}^{n}\frac{k^2}{n^2}\binom{n}{k}x^k(1-x)^{n-k}=\left(1-\frac{1}{n}\right)x^2+\frac{1}{n}x. \quad (6.167)$$

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}=\frac{x(1-x)}{n}. \quad (6.168)$$

The *Bernshtein polynomials* $B_n(f)$ of a function $f(x)$ are defined by the relationship

$$B_n(f)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right). \quad (6.169)$$

If $f(x)$ is continuous in the interval $0 \leqq x < 1$, the sequence $\{B_n(f)\}$ is uniformly convergent to $f(x)$ in this interval:

$$\lim_{n\to\infty}B_n(f)=f(x). \quad (6.170)$$

Asymptotic formulae.

$$\left(\frac{2n}{n}\right) \sim \frac{2^{2n}}{\sqrt{\pi n}} e^{\frac{\theta_n}{6n}} \quad |\theta_n| < 1. \tag{6.171}$$

$$\left(\frac{n}{m}\right) = \frac{1}{\sqrt{2\pi}} \frac{n^{n+\frac{1}{2}}}{m^{m+\frac{1}{2}}(n-m)^{n-+\frac{1}{2}}} e^{\omega}n, \ \omega_n \to 0 \text{ as } n \to \infty. \tag{6.172}$$

$$\sum_{\nu=0}^{n} \left(\frac{n}{\nu}\right)^k \sim \frac{2kn}{\sqrt{k}} \left(\frac{2}{\pi n}\right)^{\frac{k-1}{2}}. \tag{6.173}$$

$$\binom{nk+l}{n} \sim \frac{(k-1)^n}{\sqrt{2\pi n}} \left(\frac{k}{k-1}\right)^{nk+l+\frac{1}{2}}, \tag{6.174}$$

$$\binom{n+\beta}{\beta} = \frac{n^\beta}{\Gamma(\beta+1)} + \sum_{s=1}^{p} c_s n^{\beta-p} + O(n^{\beta-p-1}). \tag{6.175}$$

Fibonacci numbers and the golden section. The sums

$$u_n = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \binom{n-(k+1)}{k} \tag{6.176}$$

are connected by the relationships

$$u_n = u_{n-1} + u_{n-2}, \quad n > 2, \quad u_1 = u_2 = 1. \tag{6.177}$$

The numbers u_n are called *Fibonacci numbers. Binet's formula* holds:

$$u_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}. \tag{6.178}$$

The following are consecutive values of the Fibonacci numbers:

$$u_1, u_2, u_3, \ldots$$
$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

The Fibonacci numbers appear in the expansion as a continued fraction of the number $\frac{1}{2}(\sqrt{5}-1) = \frac{1}{1+}\frac{1}{1+\ldots}$. The convergents have the form $u_n/u_{n+1} (n = 1, 2, \ldots)$.

The number $\alpha = \frac{1}{2}(\sqrt{5}-1)$ is occasionally called the number of the *golden section.* It is defined as the mean proportional between 1 and its complement to 1:

$$\alpha^2 = 1.(1-\alpha).$$

It was supposed in classical aesthetics that, for instance, the rectangle with ratio of sides α, or the ellipse with ratio of axes α, were particularly pleasing to the eye. Hence the name golden section.

§ 2. Bernoulli and Euler numbers and polynomials

1. Bernoulli numbers and polynomials

Special number sequences, e.g. Bernoulli and Euler numbers, prove useful for obtaining power expansions of numerous functions.

1°. The *Bernoulli numbers* B_n $(n = 0, 1, 2, \ldots)$ are defined as the coefficients of the expansion as a power series of the function $t(e^t-1)^{-1}$ for $|t| < 2\pi$:

$$\frac{t}{e^t-t} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{6.179}$$

These numbers were first mentioned by Johann Bernoulli (Ars Conjectandi, 1713), in connection with the problem of summing powers of the natural numbers. The numbers B figure in the power expansions of the functions $\tan t$, $\tanh t$, $t \cot t$, t cotanh t, etc., in the Euler–Maclaurin summation formula, in the asymptotic form of Euler's gamma function.

There is no one accepted notation for the Bernoulli numbers. In certain works, B_n is used to denote what we denote in the present section by $|B_n|$, or B_{2n}, or $|B_{2n}|$.

Recurrence relations. Using (6.179), rewritten in the form

$$t = (e^t-1)\sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

cross-multiplying the series on the right-hand side and comparing coefficients of powers of t, we find that

$$C_{n+1}^{1}B_n + C_{n+1}^{2}B_{n-1} + \dots + C_{n+1}^{k}B_{n-k+1} + \dots \\ \dots + C_{n+1}^{n}B_1 + B_0 = 0 \quad (6.180)$$

or, in symbolic form:

$$(1+B)^{n+1} - B^{n+1} = 0, \quad (6.181)$$

where, after raising to a power in accordance with the binomial formula, all the exponents of the powers are replaced by subscripts.

Since

$$t(e^t-1)^{-1} - (-t)(e^{-t}-1)^{-1} = -t = \\ = 2\sum_{k=0}^{\infty} B_{2k+1} \frac{t^{2k+1}}{(2k+1)!} \quad (6.182)$$

all the Bernoulli numbers with odd subscripts, $n \geqq 3$, are zero. All the Bernoulli numbers with even subscripts are non-zero, B_{2n} being positive for n odd and negative for n even (see (6.194)).

Laplace's formula

$$B_n = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{}{(n-1)!} & \dots & \frac{1}{2!} \end{vmatrix} \quad (6.183)$$

gives an explicit expression for the value of a Bernoulli number in terms of its subscript.

The Bernoulli numbers are rational. This follows, for instance, from the previous formula.

STAUDT'S THEOREM. *Every Bernoulli number B_n can be written in the form*

$$B_n = C_n - \sum \frac{1}{k+1}, \quad (6.184)$$

where C_n is an integer, and the summation is over all $k > 0$ such that $k+1$ is prime, k being a factor of n.

For example, when $n = 6$, the factors of n are $k = 1, 2, 3, 6$. On adding unity to each of these, we get the numbers 2, 3, 4, 7, of which the primes are 2, 3, 7. Consequently, $B_6 = 1-(1/2)-(1/3)-(1/7) =$ $= 1/42$.

Some Bernoulli numbers:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30},$$

$$B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66},$$

$$B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43867}{798},$$

$$B_{20} = -\frac{174611}{330}, \quad B_{22} = \frac{854513}{123},$$

$$B_{24} = -\frac{236364091}{2730}, \quad B_{26} = \frac{8553103}{6},$$

$$B_{28} = -\frac{23749461029}{870}, \quad B_{30} = \frac{8615841276005}{14322},$$

$$B_{32} = -\frac{7709321041217}{510}, \quad B_{34} = \frac{2577687858367}{6},$$

$$B_{36} = -\frac{26315271553053477373}{1919190},$$

$$B_{38} = \frac{2929993913841559}{6}, \quad B_{40} = -\frac{261082718196449122051}{13530}.$$

Some power expansions with coefficients expressible in terms of Bernoulli numbers. Since $\frac{1}{2}\coth\frac{1}{2}t = \{t/(e^t-1)\} + \frac{1}{2}t$, it follows from (6.179), after replacing t by $2t$, that

$$t \operatorname{cotanh} t = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} t^{2n}}{(2n)!}, \quad |t| < \pi. \tag{6.185}$$

$$it \operatorname{cotanh} it = t \cot t, \quad \tan t = \cot t - 2 \cot 2t,$$

$$t \operatorname{cosec} t = t \cot t + t \tan \frac{t}{2},$$

$$\tanh t = 2 \operatorname{cotanh} 2t - \operatorname{cotanh} t,$$

$$\operatorname{cosech} t = - \operatorname{cotanh} t - \operatorname{cotanh} \frac{t}{2},$$

we get

$$t \cot t = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} t^{2n}, \quad |t| < \pi. \tag{6.186}$$

$$\tan t = \sum_{n=1}^{\infty} \frac{(2^{2n}-1)2^{2n}(-1)^{n-1}}{(2n)!} B_{2n} t^{2n-1}, \quad |t| < \frac{\pi}{2}. \tag{6.187}$$

$$t \operatorname{cosec} t = 1 + \sum_{n=1}^{\infty} \frac{(2-2^{2n})(-1)^n}{(2n)!} B_{2n} t^{2n}, \quad |t| < \pi. \tag{6.188}$$

$$\tanh t = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} t^{2n-1}, \quad |t| < \frac{\pi}{2}. \tag{6.189}$$

$$t \operatorname{cosech} t = 1 + \sum_{n=1}^{\infty} \frac{2-2^{2n}}{(2n)!} B_{2n} t^{2n-1}, \quad |t| < \pi. \tag{6.190}$$

$$\ln \frac{\sin t}{t} = \sum_{n=1}^{\infty} \frac{(-)^n 2^{2n} B_{2n}}{(2n)!\,2n} t^{2n}, \quad |t| < \pi. \tag{6.191}$$

$$\ln \cos t = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}(2^{2n}-1) B_{2n}}{(2n)!2n} t^{2n}, \quad |t| < \frac{\pi}{2}, \tag{6.192}$$

$$\ln \frac{\tan t}{t} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n}(2^{2n-1}-1) B_{2n}}{n(2n)!} t^{2n}, \quad |t| < \frac{\pi}{2}. \tag{6.193}$$

Expressions for the sums of some numerical series in terms of Bernoulli numbers. If we compare expansion (6.186) with

$$t \cot t = 1 + 2 \sum_{m=1}^{\infty} \frac{t^2}{t^2 - m^2\pi^2} = 1 - 2 \sum_{n=1}^{\infty} \frac{t^{2n}}{\pi^{2n}} \sum_{m=1}^{\infty} \frac{1}{m^{2n}}$$

it follows that

$$B_{2n} = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{m=1}^{\infty} \frac{1}{m^{2n}}. \tag{6.194}$$

The equations follow from (6.194):

$$\sum_{m=1}^{\infty} \frac{1}{m^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}. \tag{6.195}$$

$$\sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m^{2n}} = \frac{(-1)^{n-1}(2^{2n-1}-1)\pi^{2n}}{(2n)!} B_{2n}. \tag{6.196}$$

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^{2n}} = \frac{(-1)^{n-1}(2^{2n}-1)\pi^{2n}}{2(2n)!} B_{2n}. \tag{6.197}$$

In particular,

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = B_2\pi^2 = \frac{\pi^2}{6}. \tag{6.198}$$

$$\sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m^2} = \frac{B_2\pi^2}{2} = \frac{\pi^2}{12}. \tag{6.199}$$

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{3B_2\pi^2}{4} = \frac{\pi^2}{8}. \tag{6.200}$$

$$\sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{\pi^2}{90}. \tag{6.201}$$

$$\sum_{m=1}^{\infty} \frac{1}{m^6} = \frac{\pi}{945}. \tag{6.202}$$

$$\sum_{m=1}^{\infty} \frac{1}{m^8} = \frac{\pi^8}{9450}. \tag{6.203}$$

Some integrals expressible in terms of Bernoulli numbers. Integral forms of Bernoulli numbers. The relationship

$$\frac{1}{m^{2n}} = \frac{1}{(2n-1)!} \int_0^{\infty} e^{-mt} t^{2n-1} dt$$

leads in conjunction with (6.194) to the following relationships:

$$B_{2n} = (-1)^{n-1} 4n \int_0^{\infty} \frac{t^{2n-1}}{e^{2\pi t}-1} dt. \tag{6.204}$$

$$B_{2n} = (-1)^{n-1} 4\pi \int_0^{\infty} \frac{t^{2n} dt}{(e^{\pi t}-e^{-\pi t})^2}. \tag{6.205}$$

$$B_{2n} = (-1)^{n-1} \frac{2n(2n-1)}{\pi} \int_0^{\infty} t^{2n-2} \ln(1-e^{-2\pi t})\, dt. \tag{6.206}$$

$$B_{2n} = (-1)^{n-1}\left(\frac{p}{\pi}\right)^{2n}\frac{4n}{2^{2n}-1}\int_0^\infty \frac{t^{2n-1}}{\sinh pt}\,dt. \tag{6.207}$$

$$B_{2n} = (-1)^n \frac{n}{2^{2n-2}\pi^{2n}}\int_0^1 \frac{(\ln x)^{2n-1}}{1-x}\,dx. \tag{6.208}$$

$$B_{2n} = (-1)^{n-1}\frac{2n}{(1-2^{2n})\pi^{2n}}\int_0^\infty \frac{(\ln t)^{2n-1}}{1-t^2}\,dt. \tag{6.209}$$

$$B_{2n} = (-1)^{n-1}\frac{4n}{(1-2^{2n})\pi^{2n}}\int_0^1 \frac{(\ln t)^{2n-1}}{1-t^2}\,dt. \tag{6.210}$$

$$B_{2n} = (-1)^n \frac{4n}{\pi^{2n}}\int_0^1 (\ln t)^{2n-1}\frac{t\,dt}{1-t^2}. \tag{6.211}$$

$$B_{2n} = (-1)^{n-1}\frac{2}{(2^{2n}-1)\pi^{2n}}\int_0^1 (\ln t)^{2n}\frac{1+t^2}{(1-t^2)^2}\,dt; \tag{6.212}$$

$$B_{2n} = (1-)^{n-1}\frac{2n}{(1-2^{2n})\pi^{2n}}\int_0^{\pi/2} (\ln\tan t)^{2n-1}\frac{dt}{\cos 2t}. \tag{6.213}$$

$$B_{2n+2} = (-1)^{n+1}\frac{2(n+1)(2n+1)}{\pi^{2n+2}}\int_0^1 (\ln t)^{2n}\ln(1-t^2)\frac{dt}{t}. \tag{6.214}$$

2°. The *Bernoulli polynomials* are given by the formula

$$B_n(x) = (x+B)^n = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}. \tag{6.215}$$

A number of authors consider, in addition to the polynomials $B_n(x)$, the polynomials

$$\varphi_n(x) = B_n(x) - B_n. \tag{6.216}$$

It follows from (6.215) that

$$B_n(0) = B_n, \quad \varphi_n(0) = 0. \tag{6.217}$$

The generating function for the Bernoulli polynomials is $te^{xt}(e^t - 1)^{-1}$. Its expansion as a power series, convergent for $|t| < 2\pi$, has the form

$$te^{xt}(e^t-1)^{-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{6.218}$$

The difference equation

$$f(x+1) - f(x) = nx^{n-1} \tag{6.219}$$

is satisfied by the Bernoulli polynomial. This follows from (6.218) and the relationship

$$te^{(x+1)t}(e^t-1)^{-1}-te^{xt}(e^t-1)^{-1} = te^{xt}.$$

Recurrence formulae. Differentiation of (6.215) gives

$$B'_n(x) = nB_{n-1}(x) \tag{6.220}$$

from which it follows that

$$B_n(x) = B_n(0)+n\int_0^x B_{n-1}(x)\,dx. \tag{6.221}$$

Some Bernoulli polynomials:

$$B_0(x) = 1. \tag{6.222}$$

$$B_1(x) = x-\frac{1}{2}. \tag{6.223}$$

$$B_2(x) = x^2-x+\frac{1}{6}. \tag{6.224}$$

$$B_3(x) = x^3-\frac{3}{2}x^2+\frac{1}{2}x. \tag{6.225}$$

$$B_4(x) = x^4-2x^3+x^2-\frac{1}{30}. \tag{6.226}$$

$$B_5(x) = x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x. \tag{6.227}$$

$$B_6(x) = x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2+\frac{1}{42}. \tag{6.228}$$

$$B_7(x) = x^7-\frac{7}{2}x^6+\frac{7}{2}x^5-\frac{7}{6}x^3+\frac{1}{6}x. \tag{6.229}$$

$$B_8(x) = x^8-4x^7+\frac{14}{3}x^6-\frac{7}{3}x^4+\frac{2}{3}x^2-\frac{1}{30}. \tag{6.230}$$

$$B_9(x) = x^9-\frac{9}{2}x^8+6x^7-\frac{21}{5}x^5+2x^3-\frac{3}{10}x. \tag{6.231}$$

$$B_{10}(x) = x^{10}-5x^9+\frac{15}{2}x^8-7x^6+5x^4-\frac{3}{2}x^2+\frac{5}{66}. \tag{6.232}$$

Zeros of Bernoulli polynomials. The following formula holds:

$$B_n(1-x) = (-1)^n B_n(x), \tag{6.233}$$

from which, letting $x = 0$, we get

$$B_n(1) = (-1)^n B_n \tag{6.234}$$

and, since $B_{2k+1} = 0$, we have

$$B_n(1) - B_n = B_n(1) - B_n(0) = 0 \tag{6.235}$$

for all n.

It follows from (6.216) and (6.235) that

$$\varphi_n(0) = \varphi_n(1) = 0. \tag{6.236}$$

It follows from (6.233) with $n = 2k+1$ and $x = \frac{1}{2}$ that

$$B_{2k+1}\left(\frac{1}{2}\right) = -B_{2k+1}\left(\frac{1}{2}\right) \tag{6.237}$$

or

$$\varphi_{2k+1}\left(\frac{1}{2}\right) = 0. \tag{6.238}$$

The polynomials $\varphi_n(x)$ with even subscripts vanish on the segment $[0, 1]$ if and only if $x = 0$ or $x = 1$; the $\varphi_n(x)$ with odd subscripts vanish if and only if $x = 0$, $x = \frac{1}{2}$, or $x = 1$.

The polynomials $\varphi_{2k}(x)$ with even subscripts are of constant sign, the same as that of $(-1)^k$, on the interval $(0, 1)$; the polynomial $\varphi_{2k+1}(x)$ has the sign of $(-1)^{k-1}$ on the interval $\left(0, \frac{1}{2}\right)$, and the sign of $(-1)^k$ on the interval $\left(\frac{1}{2}, 1\right)$.

A multiplication theorem for Bernoulli polynomials:

$$B_n(mx) = m^{n-1} \sum_{s=0}^{m-1} B_n\left(x + \frac{s}{m}\right).$$

This relationship follows from the fact that

$$\sum_{n=0}^{\infty} \frac{B_n(mx)t^n}{n!} = \frac{e^{mxt}t}{e^t-1} = \frac{1}{m}\frac{e^{mxt}mt(1+e^t+\ldots+e^{(m-1)t})}{e^{mt}-1} =$$

$$= \frac{1}{m}\sum_{s=0}^{m-1} \frac{e^{\left(x+\frac{s}{m}\right)mt}mt}{e^{mt}-1} = \frac{1}{m}\sum_{s=0}^{m-1}\sum_{n=0}^{\infty} \frac{B_n\left(x+\frac{s}{m}\right)m^n t^n}{n!}. \tag{6.240}$$

Trigonometric expansions of Bernoulli polynomials in the interval (0, 1):

$$B_1(x) = x-\frac{1}{2} = -\sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{m}. \tag{6.241}$$

$$B_2(x) = \frac{1}{\pi^2}\sum_{m=1}^{\infty} \frac{\cos 2\pi mx}{m^2}. \tag{6.242}$$

$$\frac{(-1)^{n-1}}{(2n)!}B_{2n}(x) = \frac{1}{2^{2n-1}\pi^{2n}}\sum_{m=1}^{\infty} \frac{\cos 2\pi mx}{m^{2n}}. \tag{6.243}$$

$$\frac{(-1)^{n-1}}{(2n+1)!}B_{2n+1}(x) = \frac{1}{2^{2n}\pi^{2n+1}}\sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{m^{2n+1}}. \tag{6.244}$$

Integral forms of the Bernoulli polynomials in the interval (0, 1):

$$B_{2n}(x) = (-1)^{n+1}2n\int_0^{\infty} \frac{\cos 2\pi x - e^{-2\pi t}}{\cosh 2\pi t - \cos 2\pi x}\, t^{2n-1}\, dt \tag{6.245}$$

$$(n = 1, 2, \ldots).$$

$$B_{2n+1}(x) = (-1)^{n+1}(2n+1)\int_0^{\infty} \frac{\sin 2\pi x}{\cosh 2\pi t - \cos 2\pi x}\, t^2\, dt \tag{6.246}$$

$$(n = 0, 1, 2, \ldots).$$

Application to the summation of powers of the natural numbers. On assigning to x in (6.219) the values $0, 1, 2, \ldots, p$ and summing, we get

$$\sum_{s=1}^{p} s^{n-1} = \frac{B_n(p+1)-B_n}{n}$$

or, on replacing $n-1$ by n,

$$\sum_{s=1}^{p} s^n = \frac{B_{n+1}(p+1)-B_{n+1}}{n+1}. \tag{6.247}$$

For example,

$$\sum_{s=1}^{p} s^2 = \frac{B_3(p+1)-B_3}{3} = \frac{1}{3}\left(p^3+\frac{3}{2}p^2+\frac{p}{2}\right) = \\ = \frac{p(p+1)(2p+1)}{6}. \quad (6.248)$$

$$\sum_{s=1}^{p} s^3 = \frac{B_4(p+1)-B_4}{4} = \frac{1}{4}(p^4-2p^3+p^2) = \frac{p^2(p+1)^2}{4}. \quad (6.249)$$

$$\sum_{s=1}^{p} s^4 = \frac{B_5(p+1)-B_5}{5} = \frac{1}{5}\left(p^5-\frac{5}{2}p^4+\frac{5}{3}p^3-\frac{p}{6}\right) = \\ = \frac{1}{30}p(p+1)(2p+1)(3p^2+3p+1). \quad (6.250)$$

$$\sum_{s=1}^{p} s^5 = \frac{B_6(p+1)-B_6}{6} = \frac{1}{6}\left(p^6-3p^5+\frac{5}{2}p^4-\frac{p^2}{2}\right) = \\ = \frac{1}{12}p^2(p+1)^2(2p^2+2p+1). \quad (6.251)$$

The Euler–Maclaurin formula establishes the connection between the integrals and the sums:

$$\int_0^m f(x)\,dx = \left[\frac{f(0)+tf(m)}{2}+\sum_{k=1}^{m-1} f(k)\right] - \\ -\sum_{r=1}^{n-1}\frac{B_{2r}}{(2r)!}\,[f^{(2r-1)}(m)-f^{(2r-1)}(0)]-\frac{f^{(2n)}(\theta m)mB_{2n}}{(2n)!}. \quad (6.252)$$

If $f(x)$ and all its derivatives encountered in (6.252) tend to zero, while the derivatives of even order differ from zero and all have the same sign, the summation formula holds for $m = \infty$, if the sum and integral appearing in it are convergent.

Stirling's formula is obtained from the Euler–Maclaurin formula if we put $f(x) = \ln x$ in the latter and make the lower limit of integration equal to unity:

$$\sum_{k=1}^{m-1} \ln k = m\ln m - m - \frac{1}{2}\ln(m+1) + \\ + \sum_{r=1}^{n-1}\frac{B_{2r}(m^{-2r+1}-1)}{2r(2r-1)} - R_{n-1}. \quad (6.253)$$

When $n = 2$, it follows from (6.253) that the *asymptotic Stirling formula* for the factorial (see (6.121)) is

$$m! \approx \sqrt{2\pi m}\, m^{m+\frac{1}{2}} e^{-m}. \tag{6.254}$$

2. Euler numbers and polynomials

1°. The *Euler numbers* E_k ($k = 0, 1, 2, \ldots$) are defined as the coefficients in the expansion as a power series of the function

$$\operatorname{sech} t = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}, \quad |t| < \frac{\pi}{2}. \tag{6.255}$$

Since the function $\operatorname{sech} t = 2/(e^t + e^{-t})$ is even, it follows from this that the Euler numbers with odd subscripts are zero:

$$E_{2m+1} = 0 \quad (m = 1, 2, \ldots). \tag{6.256}$$

Recurrence relations. It follows from (6.255) that

$$1 = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}, \tag{6.257}$$

whence $E_0 = 1$ and

$$E_0 + C_{2n}^2 E_2 + C_{2n}^4 E_4 + \ldots + C_{2n}^{2n-2} E_{2n-2} + E_{2n} = 0. \tag{6.258}$$

In symbolic form:

$$(E+1)^n + (E-1)^n = 0, \quad E_0 = 1. \tag{6.259}$$

Some Euler numbers:

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$$
$$E_{10} = -50251, \quad E_{12} = 2702765$$
$$E_{14} = -199360981, \quad E_{16} = 19391512145,$$
$$E_{18} = -2404879675441, \quad E_{20} = 370371188237525.$$

All the Euler numbers are integers. This follows from the recurrence formulae for the E_k. Two adjacent numbers with even subscripts have opposite signs.

SYLVESTER'S THEOREM. *If α, β, γ, ... are factors of the number $n-m$, the difference $E_{2n} - E_{2m}$ is divisible by those of the numbers $2\alpha+1$, $2\beta+1$, $2\gamma+1$, ... that are prime.*

The connection with Bernoulli numbers. Starting from the identity

$$\frac{4x}{e^{4x}-1}-\frac{2x}{e^{2x}-1}=-\frac{2xe^{-x}}{e^{x}+e^{-x}},$$

we obtain

$$\sum_{n=0}^{\infty} B_n 2^n(2^n-1)\frac{x^n}{n!} = -\sum_{l=0}^{\infty} E_l \frac{x^l}{l!}\sum_{m=0}^{\infty}\frac{(-1)^m x^{m+1}}{m!},$$

from which it follows that

$$B_n = -\frac{n(E-1)^{n-1}}{2^n(2^n-1)}. \tag{6.260}$$

Starting from the identity

$$\frac{4x}{e^{4x}-1}(e^{3x}-e^{x}) = \frac{4x}{e^{x}+e^{-x}},$$

we get

$$\sum_{m=0}^{\infty} B_m \frac{(4x)^m}{m!}\sum_{l=0}^{\infty}\left[\frac{(3x)^l}{l!}-\frac{x^l}{l!}\right] = 2x\sum_{k=0}^{\infty} E_k\frac{x^k}{k!},$$

which gives

$$E_{k-1} = \frac{(4B+3)^k-(4B+1)^k}{2k}. \tag{6.261}$$

It follows from the last relationship that

$$\frac{2k+1}{24k+1}E_{2k} = \left(B+\frac{3}{4}\right)^{2k+1}-\left(B+\frac{1}{4}\right)^{2k+1}, \tag{6.262}$$

which gives, on taking (6.215) and (6.233) into account,

$$\frac{2k+1}{24k+2}E_{2k} = -2B_{2k+1}\left(\frac{1}{4}\right). \tag{6.263}$$

The sign of the Euler numbers. As follows from the discussion on page 329, $B_{2k+1}(x)$ has the sign of $(-1)^{k-1}$ in the interval (0, 1/2), so that E_{2k} has the sign of $(-1)^k$.

Some power expansions with coefficients expressible in terms of Euler numbers. Observing that $\sec t = \operatorname{sech} it$, we get

$$\sec t = \sum_{k=0}^{\infty}(-1)^k\frac{E_{2k}}{(2k)!}t^{2k}, \quad |t| < \frac{\pi}{2}. \tag{6.264}$$

On integrating the last series, we get

$$\ln\left|\tan\left(\frac{\pi}{4}+\frac{t}{2}\right)\right| = \sum_{k=0}^{\infty}(-1)^k\frac{E_{2k}}{(2k+1)!}t^{2k+1},\ |t|<\frac{\pi}{2}. \qquad (6.265)$$

Integration of series (6.255) gives

$$\operatorname{arc\,tan} e^t = \frac{\pi}{4}+\sum_{k=0}^{\infty}\frac{1}{2}E_k\frac{t^{k+1}}{(k+1)!}. \qquad (6.266)$$

The expressions for the sums of certain numerical series in terms of Euler numbers. On putting $x = 1/4$ in (6.244) and taking (6.263) into account, we get

$$1-\frac{1}{3^{2k+1}}+\frac{1}{5^{2k+1}}-\frac{1}{7^{2k+1}}+\ldots = \frac{(-1)^kE_{2k}\pi^{2k+1}}{4^{k+1}(2k)!}. \qquad (6.267)$$

For example,

$$\sum_{m=0}^{\infty}(-1)^m\frac{1}{2m+1} = \frac{\pi}{4},\quad \sum_{m=0}^{\infty}(-1)^m\frac{1}{(2m+1)^3} = \frac{\pi^3}{32}. \qquad (6.268)$$

Integral forms of Euler numbers. It follows from (6.246) and (6.263) that

$$E_{2n} = (-1)^n2^{2k+1}\int_0^{\infty}t^{2n}\operatorname{sech}(\pi t)dt \quad (n = 0,\ 1,\ 2,\ \ldots). \qquad (6.269)$$

2°. The *Euler polynomials* are given by

$$E_n(x) = \left(x-\frac{1}{2}+\frac{E}{2}\right)^n = \sum_{k=0}^{n}\binom{n}{k}2^{-k}E_k\left(x-\frac{1}{2}\right)^{n-k}, \qquad (6.270)$$

where E_k are Euler numbers.

The generating function for the Euler polynomials is $2e^{xt}(e^t+1)^{-1}$. Its expansion as a power series, convergent for $|t| < \pi$, is

$$2e^{xt}(e^t+1)^{-1} = \sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!}. \qquad (6.271)$$

The difference equation

$$f(x+1)+f(x) = 2x^n \qquad (6.272)$$

is satisfied by the Euler polynomial $E_n(x)$. This follows from the relationship

$$2e^{(x+1)t}(e^t+1)^{-1}+2e^{xt}(e^t+1)^{-1} = 2e^{xt}.$$

Recurrence formulae. On differentiating (6.270), we get

$$E_n'(x) = nE_{n-1}(x) \tag{6.273}$$

from which it follows that

$$e_n(x) = E_n\left(\frac{1}{2}\right)+n\int_{1/2}^{x} E_{n-1}(x)\,dx. \tag{6.274}$$

Since $E_n(1/2) = 2^{-n}E_n$ (see (6.270)), we have

$$E_n(x) = 2^{-n}E_n+n\int_{1/2}^{x} E_{n-1}(x)\,dx. \tag{6.275}$$

In view of the relationship

$$2e^{(x+1)t}(e^t+1)^{-1} = \sum_{r=0}^{\infty} E_r(x)\frac{t^r}{r!}\sum_{m=0}^{\infty}\frac{t^m}{m!} = \sum_{n=0}^{\infty} E_n(x+1)\frac{t^n}{n!}$$

we have the recurrence formula

$$E_n(x+1) = \sum_{r=0}^{\infty}\left(\frac{n}{r}\right)E_r(x), \tag{6.276}$$

or, on taking (6.272) into account,

$$E_n(x) = 2x^n-\sum_{r=0}^{n}\binom{n}{r}E_r(x). \tag{6.277}$$

Some Euler polynomials:

$$E_0(x) = 1. \tag{6.278}$$

$$E_1(x) = x-\frac{1}{2}. \tag{6.279}$$

$$E_2(x) = x^2-x. \tag{6.280}$$

$$E_3(x) = x^3-\frac{3}{2}x^2+\frac{1}{4}. \tag{6.281}$$

$$E_4(x) = x^4-2x^3+x. \tag{6.282}$$

$$E_5(x) = x^5-\frac{5}{2}x^4+\frac{5}{2}x^2-\frac{1}{2}. \tag{6.283}$$

$$E_6(x) = x^6-3x^5+5x^3-3x. \tag{6.284}$$

$$E_7(x) = x^7-\frac{7}{2}x^6+\frac{35}{4}x^4-\frac{21}{2}x^2+\frac{17}{8}. \tag{6.285}$$

$$E_8(x) = x^8-4x^7+14x^5-28x^3-17x. \tag{6.286}$$

The connection between Euler and Bernoulli polynomials. Starting from the relationship

$$te^{\frac{xt}{2}}\left(e^{\frac{t}{2}}+1\right)^{-1} = te^{(x+1)\frac{t}{2}}(e^t-1)^{-1}-te^{\frac{xt}{2}}(e^t-1)^{-1},$$

t—

we get

$$E_{n-1}(x) = n^{-1}2^n\left\{B_n\left(\frac{x+1}{2}\right)-B_n\left(\frac{x}{2}\right)\right\} =$$

$$= n^{-1}2\left\{B_n(x)-2^nB_n\left(\frac{x}{2}\right)\right\}. \quad (6.287)$$

Trigonometric expansions of Euler polynomials. On using the trigonometric expansions of Bernoulli polynomials in the interval $0 < x < 1$, we get

$$E_{2k}(x) = (-1)^k\,4(2k)!\sum_{n=0}^{\infty}[(2n+1)\pi]^{-2k-1}\sin(2n+1)\pi x$$

$$(n = 1, 2, 3, \ldots). \quad (6.288)$$

$$E_{2k+1}(x) = (-1)^{k+1}\,4(2k+1)!\sum_{n=0}^{\infty}[(2n+1)\pi]^{-2k-2}\cos(2n+1)\pi x$$

$$(n = 0, 1, 2, 3, \ldots). \quad (6.289)$$

A multiplication theorem:

$$E_n(mx) = m^n\sum_{r=0}^{m-1}(-1)^rE_n\left(x+\frac{r}{m}\right), \qquad m = 2k+1. \quad (6.290)$$

$$E_n(mx) = -2m^n(n+1)^{-1}\sum_{r=0}^{m-1}(-1)^rB_{n+1}\left(x+\frac{r}{m}\right), \quad m = 2k. \quad (6.291)$$

Integral forms of Euler polynomials may be obtained in the same way as those of Bernoulli polynomials:

$$E_{2n}(x) = (-1)^n\,4\int_0^{\infty}\frac{t^{2n}\sin\pi x\cosh\pi t}{\cosh 2\pi t-\cos 2\pi x}\,dt \quad (6.292)$$

$$(0 < x < 1;\quad n = 0, 1, 2, \ldots),$$

$$E_{2n+1}(x) = (-1)^{n+1}\,4\int_0^{\infty}\frac{t^{2n+1}\cos\pi x\sinh\pi t}{\operatorname{csoh} 2\pi t-\cos 2\pi x}\,dt \quad (6.293)$$

$$(0 < x < 1;\quad n = 0, 1, 2, \ldots).$$

§ 3. Elementary piecewise linear functions and delta-shaped functions

1. Piecewise linear functions

1°. The absolute value of x (written as $|x|$) is:

$$|x| = \begin{cases} -x, & x < 0, \\ 0, & x = 0, \\ x, & x > 0. \end{cases} \tag{6.294}$$

$$|x+y| \leqq |x|+|y|, \quad |x-y| \geqq ||x|-|y||. \tag{6.295}$$

The binomial expansion of $|x| = \sqrt{1-(1-x^2)}$, for $|x| < 1$, is

$$|x| = 1-\frac{1-x^2}{2}-\sum_{k=2}^{\infty}\frac{(2k-3)!!}{(2k)!!}(1-x^2)^k. \tag{6.296}$$

The approximation, for $|x| \leqq 1$, by Bernshtein polynomials (see § 1, sec. 2) of even degree is

$$|x| = \frac{1}{4}\lim_{n\to\infty}\left(\frac{1-x^2}{4}\right)^n\sum_{k=1}^{n} k\binom{2n}{n-k}\left[\left(\frac{1+x}{1-x}\right)^k+\left(\frac{1-x}{1+x}\right)^k\right]. \tag{6.297}$$

The approximation by Fejer polynomials, for $|x| \leqq \pi$, is

$$|x| = \lim_{n\to\infty}\sigma_{2n-1} = \lim_{n\to\infty}\left[\frac{\pi}{2}-\frac{8}{\pi}\sum_{k=1}^{n-1}\frac{n-k}{2n-1}\,\frac{\cos(2k-1)x}{(2k-1)^2}\right]. \tag{6.298}$$

The expansion in Legendre polynomials $P_n(x)$ (see Chapter IV, §4, sec. 5), for $|x| < 1$, is

$$\left.\begin{aligned} |x| &= \sum_{n=0}^{\infty} a_n P_n(x), \\ a_{2k+1} = 0,\ a_0 = \frac{1}{2},\ a_2 &= \frac{5}{8},\ a_{2k}=(-1)^{k+1}(4k+1)\frac{(2k-3)!!}{(2k+2)!!}. \end{aligned}\right\} \tag{6.299}$$

Fourier's expansion, for $|x| < \pi$, is

$$|x| = \frac{\pi}{4}-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}. \tag{6.300}$$

The integral form is

$$|x| = \frac{2}{\pi}\int_0^{\infty} \frac{1-\cos xt}{t^2}\,dt. \tag{6.301}$$

Polygonal functions; the approximation of continuous functions by polygonal functions. Let $A_i(x_i, y_i)$ $(i = 0, 1, 2, \ldots, n, \ldots)$ be given points. The function

$$f_n(x) = y_0+\frac{k_1}{2}[x-x_0+|x-x_0|]+$$
$$+\frac{1}{2}\sum_{i=1}^{n-1}(k_{i+1}-k_i)[(x-x_i)+|x-x_i|], \tag{6.302}$$

where $k_i = (y_{i+1}-y_i)/(x_{i+1}-x_i)$, is described as *polygonal;* its graph is the polygon $A_0A_1A_2 \ldots A_n$. It must be borne in mind that

$$\frac{1}{2}[x-\alpha+|x-\alpha|] = \begin{cases} 0 & \text{for } x \leqq \alpha, \\ x-\alpha & \text{for } x > \alpha. \end{cases}$$

If $f(x)$ is a continuous function in $[a, b]$, there corresponds to the subdivision $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ a polygona function whose graph is a polygon inscribed in the graph of $f(x)$, with vertices at the points (x_i, y_i), $y_i = f(x_i)$ $(i = 0, 1, 2, \ldots, n)$.

Every continuous function is the uniform limit of polygonal functions.

On taking expansions (6.296)–(6.301) into account, we find that any polygonal function, defined in an arbitrary segment $[a, b]$, is the limit of a polynomial sequence, uniformly convergent in $[a, b]$ (*Weierstrass's theorem*).

Schauder's basis. The polygonal function

$$F_{\alpha\beta}(x) = \frac{1}{\beta-\alpha}(|x-\alpha|-|2x-\alpha-\beta|+|x-\beta|),\ \alpha < \beta, \tag{6.303}$$

is equal to zero outside the interval (α, β); in (α, β) its graph is an isosceles triangle of height 1:

$$\begin{aligned} F_{\alpha\beta}(x) &= 0,\ x < \alpha,\ x > \beta; \\ F_{c\beta}(x) &= 2x-2\alpha,\ x < \frac{\alpha+\beta}{2}; \\ F_{\alpha\beta}(x) &= -2x+2\beta,\ x > \frac{\alpha+\beta}{2}. \end{aligned} \tag{6.304}$$

Let $y = f(x)$ be a continuous function in (α, β). We introduce the numbers

$$d_{\alpha\beta}(f) = f\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{2}[f(\alpha)+f(\beta)]. \tag{6.305}$$

Schauder's basis is defined as the sequence of functions in $[0, 1]$:

$$\begin{gathered} 1,\ x,\ F_{01}(x),\ F_{0,\frac{1}{2}}(x),\ F_{\frac{1}{2},1}(x),\ F_{0,\frac{1}{4}}(x), \\ F_{\frac{1}{4},\frac{1}{2}}(x),\ \ldots,\ F_{\frac{i}{2k},\frac{i+1}{2k}}(x),\ \ldots \\ (i = 0, 1, \ldots, 2^k-1;\ \ k = 0, 1, 2, \ldots). \end{gathered} \tag{6.306}$$

Any function $f(x)$, continuous in the interval $[0, 1]$, can be expanded (uniquely) as the series

$$\begin{aligned} f(x) = {} & f(0)+[f(1)-f(0)]x+ \\ & + \sum_{k=0}^{\infty} \sum_{i=0}^{2^k-1} d_{\frac{i}{2^k},\frac{i+1}{2^k}}(f)\, F_{\frac{i}{2^k},\frac{i+1}{2^k}}(x). \end{aligned} \tag{6.307}$$

The partial sum of the last series

$$\begin{aligned} s_n(x) = {} & f(0)+[f(1)-f(0)]\,x+ \\ & + \sum_{k=0}^{n} \sum_{i=0}^{2^k-1} d_{\frac{i}{2^k},\frac{i+1}{2^k}}(f)\, F_{\frac{i}{2^k},\frac{i+1}{2^k}}(x) \end{aligned} \tag{6.308}$$

has a graph consisting of a step-function, inscribed in the curve $y = f(x)$, with vertices at the points $(i/2^n, f(i/2^n))$ $(i = 0, 1, \ldots, 2^n)$.

The derivatives of the functions of Schauder's basis form a *system of Haar functions* (see (4.24) and (4.25)).

2°. Sign x (signature x) is

$$\operatorname{sign} x = \begin{cases} -1 & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x = 0, \\ 1 & \text{for} \quad x > 0 \end{cases} \tag{6.309}$$

or

$$\operatorname{sign} x = \begin{cases} \dfrac{x}{|x|} & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0 \end{cases} \tag{6.310}$$

or

$$\operatorname{sign} x = \lim_{n\to\infty} \frac{2}{\pi} \operatorname{arc\,tan} nx. \qquad (6.311)$$

Fourier's expansion is

$$\operatorname{sign} x = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin (2k+1)x}{2k+1}, \qquad |x| < \pi. \qquad (6.312)$$

The expansion in Hermitian polynomials (see Chapter IV, § 4, sec. 10) is

$$\operatorname{sign} x = \sum_{n=1}^{\infty} (-1)^n \frac{(2n-2)!}{\sqrt{2\pi}2^{n-1}} \frac{2n}{n!} H_{2n+1}(x). \qquad (6.313)$$

The integral form is

$$\operatorname{sign} x = \lim_{T\to\infty} \frac{2}{\pi} \int_0^{xT} \frac{\sin t}{t}\, dt. \qquad (6.314)$$

The sequence of signs of sine (*Rademacher functions*) is

$$p_k(t) = \operatorname{sign} (\sin 2k\pi t), \qquad (6.315)$$

$$r_k(t) = \operatorname{sign} (\sin 2k\pi t). \qquad (6.316)$$

3°. The integral part x (written as $[x]$, entière x, $E(x)$). If $x = n+r$, n is an integer, $0 \leqq r < 1$, then $[x] = n$.

The following relationships hold:

$$[x+y] \geqq [x]+[y]; \qquad (6.317)$$

$$\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right], \quad n \text{ is an integer}; \qquad (6.318)$$

$$[x]+\left[x+\frac{1}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right] = [nx]. \qquad (6.319)$$

If p and q are mutually prime integers, we have

$$\left[\frac{p}{q}\right]+\left[\frac{2p}{q}\right]+\left[\frac{3p}{q}\right]+\ldots+\left[\frac{(q-1)p}{q}\right] = \frac{(p-1)(q-1)}{2}; \qquad (6.320)$$

$$\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\ldots+\left[\frac{n}{k}\right]+\ldots+\left[\frac{n}{n}\right] =$$
$$= s_1+s_2+\ldots+s_k+\ldots+s_n, \qquad (6.321)$$

where s_k is the number of factors of k.

Fourier's expansion is

$$x-[x] = \frac{1}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n}\,(x \neq 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots). \quad (6.322)$$

The function $\{x\} = x-[x]$ is called the *fractional part of* x. It is a periodic function with period equal to unity.

4°. The distance to the nearest integer [denoted by (x)] is

$$(x) = \min\{x-[x],\ 1+[x]-x\}\,. \quad (6.323)$$

The function (x) is periodic with period equal to unity.

The Fourier expansion is

$$(x) = \frac{1}{4}-\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos 2\pi(n+1)x}{(2n+1)^2}\,. \quad (6.324)$$

5°. The jump function $1(x)$ (*Heaviside's unit function*) is

$$1(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad x \geqq 0. \end{cases} \quad (6.325)$$

The rectangular pulse is

$$1(x-\beta) = 1(x-\alpha) = \begin{cases} 0 & \text{for} \quad x < \alpha, \quad x > \beta, \\ 1 & \text{for} \quad \alpha \leqq x \leqq \beta, \end{cases} \qquad \beta > \alpha. \quad (6.326)$$

If $f(x)$ is a continuous function in $[a, b]$, the sequence of functions

$$f_n(x) = \sum_{k=1}^{n} f(x_k)[1(x-x_k)-1(x-x_{k-1})] \quad (6.327)$$

tends uniformly to $f(x)$ on indefinite subdivision of $[a, b]$ by the points x_k:

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

6°. The representation of certain piecewise linear functions with the aid of integrals and series:

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin \xi a}{\xi}\cos \xi x\, d\xi = \begin{cases} 1 & \text{for} \quad 0 < x < a, \\ 0 & \text{for} \quad x \geqq a. \end{cases} \quad (6.328)$$

$$\int_0^{\infty} aJ_1(a\xi)J_0(\xi x)d\xi = \begin{cases} 1 & \text{for} \quad 0 < x < a, \\ 0 & \text{for} \quad x > a. \end{cases} \quad (6.329)$$

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{2a}{\pi n}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{a}=\begin{cases}+1 & \text{for}\quad 0<x<\frac{1}{2}a,\\ -1 & \text{for}\quad \frac{1}{2}a<x<a.\end{cases}\tag{6.330}$$

$$\frac{2}{a}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a^2}{\pi n}\sin\frac{n\pi x}{a}=x,\qquad 0\leqq x\leqq a.\tag{6.331}$$

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{a}{\pi n}\sin\frac{n\pi x}{a}=1-\frac{x}{a},\qquad 0\leqq x\leqq a.\tag{6.332}$$

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{2a^2}{\pi^2n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{a}=\begin{cases}x & \text{for}\quad 0\leqq x\leqq\frac{1}{2}a.\\ a-x & \text{for}\quad \frac{1}{2}a\leqq x\leqq a.\end{cases}\tag{6.333}$$

$$2\int_0^{\infty}\frac{\cos\frac{1-\theta}{\theta}u-\cos\frac{u}{\theta}}{\pi u^2}\cos\frac{ut}{\theta}\,du=$$

$$=\begin{cases}1 & \text{for}\quad |t|\leqq 1-\theta,\\ \frac{1}{\theta}(1-|t|) & \text{for}\quad 1-\theta\leqq|t|\leqq 1.\\ 0 & \text{for}\quad |t|\geqq 1.\end{cases}\tag{6.334}$$

$$2\int_{-\infty}^{\infty}\frac{\sin^2\frac{x}{2}}{\pi x^2}\cos tx\,dx=\begin{cases}1-|t| & \text{for}\quad |t|\leqq 1,\\ 0 & \text{for}\quad |t|\geqq 1.\end{cases}\tag{6.335}$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin az}{z}\cos zx\,dz=\begin{cases}1 & \text{for}\quad 0\leqq x<a,\\ \frac{1}{2} & \text{for}\quad x=a,\\ 0 & \text{for}\quad x>a.\end{cases}\tag{6.336}$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin\alpha x}{x}dx=\begin{cases}-1 & \text{for}\quad \alpha<0,\\ 0 & \text{for}\quad \alpha=0,\\ 1 & \text{for}\quad \alpha>0.\end{cases}\tag{6.337}$$

$$\lim_{T\to\infty}\frac{2}{\pi}\int_0^T\frac{\sin ht}{t}\cos(x-a)t\,dt=\begin{cases}0 & \text{for } x<a-h,\\ \dfrac{1}{2} & \text{for } x=a-h,\\ 1 & \text{for } a-h<x<a+h,\\ \dfrac{1}{2} & \text{for } x=a+h,\\ 0 & \text{for } x>a+h.\end{cases} \tag{6.338}$$

2. The δ (delta)-function

A sequence of functions $\{u_\nu(x)\}$ is said to be *weakly convergent* in the interval (a, b) if, given any continuous function $f(x)$, the following limit exists:

$$\lim_{\nu\to\infty}\int_a^b f(x)u_\nu(x)dx. \tag{6.339}$$

If the sequence $\{u_\nu(x)\}$ satisfies the conditions:

I. For any $M>0$, we have for $|a|\leqq M$, $|b|\leqq M$:

$$\left|\int_a^b u_\nu(x)\,dx\right|<K(M); \tag{6.340}$$

II. Given fixed a and b, $|a|>0$, $|b|>0$,

$$\lim_{\nu\to\infty}\int_a^n u_\nu(x)\,dx=\begin{cases}0 & \text{for } a<b<0 \text{ or } 0<a<b,\\ 1 & \text{for } a<0<b.\end{cases} \tag{6.341}$$

then the limit (6.339) exists, does not depend on the choice of sequence $\{u_\nu(x)\}$ and is equal to $f(0)$. The limiting element of the (weakly) convergent sequence $\{u_\nu(x)\}$ is referred to here as a *delta function*: $\delta(x)$. Thus

$$\int_a^b f(x)\delta(x)\,dx=f(0). \tag{6.342}$$

Sequences satisfying conditions I and II are described as *delta-shaped.*

Examples of delta-shaped sequences are:

$$u_\varepsilon(x)=\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2},\quad \varepsilon>0,\quad u_\varepsilon(x)\to\delta(x); \tag{6.343}$$

$$u_t(x) = \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{x^2}{4t}}, \quad t > 0, \quad u_t(x) \to \delta(x); \tag{6.344}$$

$$u_\nu(x) = \frac{1}{\pi}\frac{\sin \nu x}{x}, \quad 0 < \nu < \infty, \quad u_\nu(x) \to \delta(x). \tag{6.345}$$

The elementary properties of the delta function are:

$$\delta(-x) = \delta(x); \tag{6.346}$$

$$f(x')\delta(x'-x) = f(x)\delta(x'-x); \tag{6.347}$$

$$x\delta(x) = 0; \tag{6.348}$$

$$\delta(\varphi(x)) = \sum_s \frac{\delta(x-x_s)}{|\varphi'(x)|} = \sum_s \frac{\delta(x-x_s)}{|\varphi'(x_s)|}, \tag{6.349}$$

where x_s are the simple roots of the equation $\varphi(x) = 0$;

$$\delta(ax) = \frac{\delta(x)}{|a|}; \tag{6.350}$$

$$\delta(x^2-a^2) = \frac{\delta(x-a)+\delta(x+a)}{2|x|}; \tag{6.351}$$

$$|x|\delta(x^2) = \delta(x). \tag{6.352}$$

The convergence of the series expansions and the integrals quoted below is understood as weak convergence.

The Fourier expansion, $|x| \leqq 1$, is:

$$\delta(x-x') = \frac{1}{2l} + \frac{1}{l}\sum_{n=1}^{\infty} \cos \frac{n\pi(x-x')}{l}. \tag{6.353}$$

The expansion in Bessel functions (see § 4, sec. 6), $0 \leqq r \leqq R$, is:

$$\delta(r'-r) = \sum_{n=0}^{\infty} \frac{2\sqrt{rr'}}{R^2} \frac{J_m\left(s_n \frac{r'}{R}\right) J_m\left(s_n \frac{r}{R}\right)}{J'_m(s_n)}. \tag{6.354}$$

The expansion in Legendre polynomials, $|x| \leqq 1$, is:

$$\delta(x'-x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x')P_n(x). \tag{6.355}$$

The expansion in Hermite polynomials is:

$$\delta(x'-x)\sum_{n=0}^{\infty}\frac{1}{\sqrt{\pi 2^n n!}}e^{-\frac{x^2x'^2}{2}}H_n(x')H_n(x). \tag{6.356}$$

Integral forms. Fourier's integral is:

$$\delta(x'-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos k(x'-x)\,dk. \tag{6.357}$$

The Fourier–Bessel integral is:

$$\left.\begin{aligned}\delta(r'-r) &= \sqrt{rr'}\int_0^{\infty}J_m(kr)J_m(kr')\,dk,\\ \delta(r') &= r'\int_0^{\infty}kJ_0(kr')\,dk.\end{aligned}\right\} \tag{6.358}$$

The involution is:

$$\int_{-\infty}^{\infty}\delta(t-\mu_1)\delta(t-\mu_2)dt = \delta(\mu_1-\mu_2). \tag{6.359}$$

The derivatives of the delta function are given by:

$$\delta(x) = 1'(x); \tag{6.360}$$

$$\int_{-\infty}^{\infty}f(x)\delta'(x-h)\,dx = -f'(h); \tag{6.361}$$

$$\int_{-\infty}^{\infty}f(x)\delta^{(k)}(x-h)dx = (-1)^k f^{(k)}(h). \tag{6.362}$$

Series of delta functions are:

$$\cos x+\cos 2x+\dots+\cos nx+\dots = \\ = -\frac{1}{2}+\pi\sum_{-\infty}^{\infty}\delta(x-2\pi n); \tag{6.363}$$

$$\sin x+2\sin 2x+\dots+n\sin nx+\dots = \\ = -\pi\sum_{-\infty}^{\infty}\delta'(x-2\pi n); \tag{6.364}$$

$$\cos x+4\cos 2x+\dots+n^2\cos nx+\dots = \\ = -\pi\sum_{-\infty}^{\infty}\delta''(x-2\pi n). \tag{6.365}$$

§ 4. Elementary special functions

1. Elliptic integrals

Definition. Any integral of the form

$$\int_0^x \frac{R(x)}{\sqrt{P(x)}}\,dx,$$

where $R(x)$ is a rational function of x, and $P(x)$ is a polynomial of the third or fourth degree, can be transformed to a sum of integrals, reducible to elementary functions, and to integrals known as *elliptic integrals in Legendre's normal form*. The latter include the following:

the elliptic integral of the first kind,

$$F(k, \varphi) = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}; \tag{6.366}$$

the elliptic integral of the second kind,

$$E(k, \varphi) = \int_0^{\sin\varphi} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx; \tag{6.367}$$

the elliptic integral of the third kind,

$$\Pi(k, \lambda, \varphi) = \int_0^{\sin\varphi} \frac{dx}{(1+\lambda x^2)\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{6.368}$$

The notations: $F(\varphi, k)$, $E(\varphi, k)$ and $\Pi(\varphi, \lambda, k)$ are sometimes employed.

The parameter k is called the *modulus* of the integral, and λ the *parameter* of the integral of the third kind, $k^2 < 1$.

The number $k' = \sqrt{1-k^2}$ is called the *complementary modulus.*

The quantity α is often employed, instead of k, where $k = \sin\alpha$.

If we put $x = \sin\psi$, the elliptic integrals reduce to the *normal trigonometric form*:

$$F(k, \varphi) = \int_0^{\varphi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = \int_0^{\varphi} \frac{d\psi}{\Delta\psi}, \tag{6.369}$$

$$E(k, \varphi) = \int_0^{\varphi} \sqrt{1-k^2 \sin^2 \psi}\, d\psi = \int_0^{\varphi} \Delta\psi\, d\psi, \tag{6.370}$$

$$\Pi(k, \lambda, \varphi) = \int_0^{\varphi} \frac{d\psi}{(1+\lambda \sin^2 \psi)\sqrt{1-k^2 \sin^2 \psi}} = \int_0^{\varphi} \frac{d\psi}{(1+\lambda \sin^2 \psi)\Delta\psi}, \tag{6.371}$$

where $\Delta\psi = \sqrt{1-k^2 \sin^2 \psi}$.

The following notation is used for a commonly encountered combination of these integrals:

$$D(k, \varphi) = \frac{F(k, \varphi)-E(k, \varphi)}{k^2} = \int_0^{\varphi} \frac{\sin^2 \psi}{\Delta\psi}\, d\psi = $$

$$= \int_0^{\sin \varphi} \frac{x^2\, dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{6.372}$$

Representations as series:

$$F(k, \varphi) = A_0 + \frac{1}{2} A_1 k^2 + \frac{1.3}{2.4} A_2 k^4 + \frac{1.3.5}{2.4.6} A_3 k^6 + \ldots. \tag{6.373}$$

$$E(k, \varphi) = A_0 - \frac{1}{2} A_1 k^2 - \frac{1}{2.4} A_2 k^4 - \frac{1.3}{2.4.6} A_3 k^6 - $$

$$- \frac{1.3.5}{2.4.6.8} A_4 k^8 - \ldots, \tag{6.374}$$

where

$$A_n = \int_0^{\varphi} \sin^{2n}\alpha\, d\alpha \qquad (n = 0, 1, 2, \ldots). \tag{6.375}$$

The coefficients A_n can be evaluated successively in accordance with the formulae:

$$A_0 = \varphi,\ A_n = \frac{2n-1}{2n} A_{n-1} \frac{1}{2} \cos \varphi \sin^{2n-1} \varphi. \tag{6.376}$$

If φ is close to $\pi/2$, the following formulae are convenient:

$$F(k, \varphi) = F\left(k, \frac{\pi}{2}\right) - F(k, \varphi_1). \tag{6.377}$$

$$E(k,\ \varphi) = E\left(k,\ \frac{\pi}{2}\right) - E(k,\ \varphi_1) + k^2 \sin\varphi \sin\varphi_1, \tag{6.378}$$

φ_1 being given by the equation

$$\sin\varphi_1 = \frac{\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}. \tag{6.379}$$

If the upper limit of the integrals is equal to $\pi/2$ ($\varphi = \pi/2$), the *complete elliptic integrals* are obtained.

The *complete elliptic integral of the first kind* is

$$\boldsymbol{K} = F\left(k,\ \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\varphi}{\Delta\varphi} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{6.380}$$

The *complete elliptic integral of the second kind* is

$$\boldsymbol{E} = E\left(k,\ \frac{\pi}{2}\right) = \int_0^{\pi/2} \Delta\varphi d\varphi = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\, dx. \tag{6.381}$$

Here,

$$\boldsymbol{D} = D\left(k,\ \frac{\pi}{2}\right) = \frac{\boldsymbol{K}-\boldsymbol{E}}{k^2}. \tag{6.382}$$

Representations as series and products:

$$\boldsymbol{K} = \frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}\right\}; \tag{6.383}$$

$$\boldsymbol{E} = \frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty} (-1)^n \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1}\right\}; \tag{6.384}$$

$$\boldsymbol{D} = \pi \sum_{n=1}^{\infty} \frac{n}{2n-1} \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n-2}; \tag{6.385}$$

$$\boldsymbol{K} = \frac{\pi}{2} \prod_{n=1}^{\infty} (1+k_n); \quad \text{where} \quad k_n = \frac{1-\sqrt{1-k_{n-1}^2}}{1+\sqrt{1-k_{n-1}^2}}, \quad k_0 = k. \tag{6.386}$$

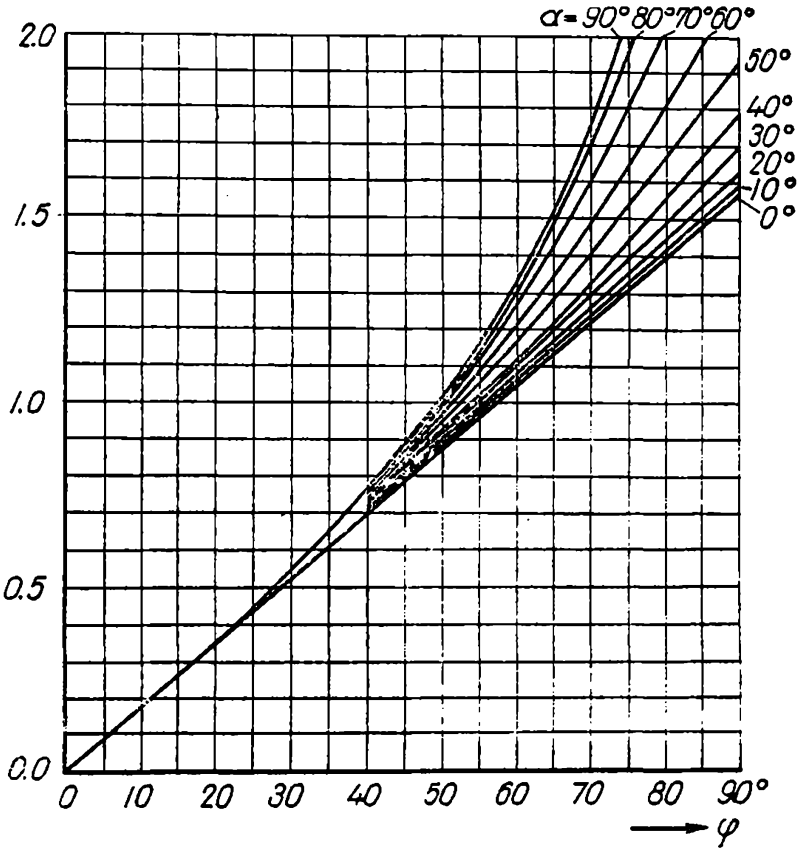


FIG. 5. Elliptic integral of the first kind $F(k, \varphi)$; $k = \sin\alpha = \text{const.}$

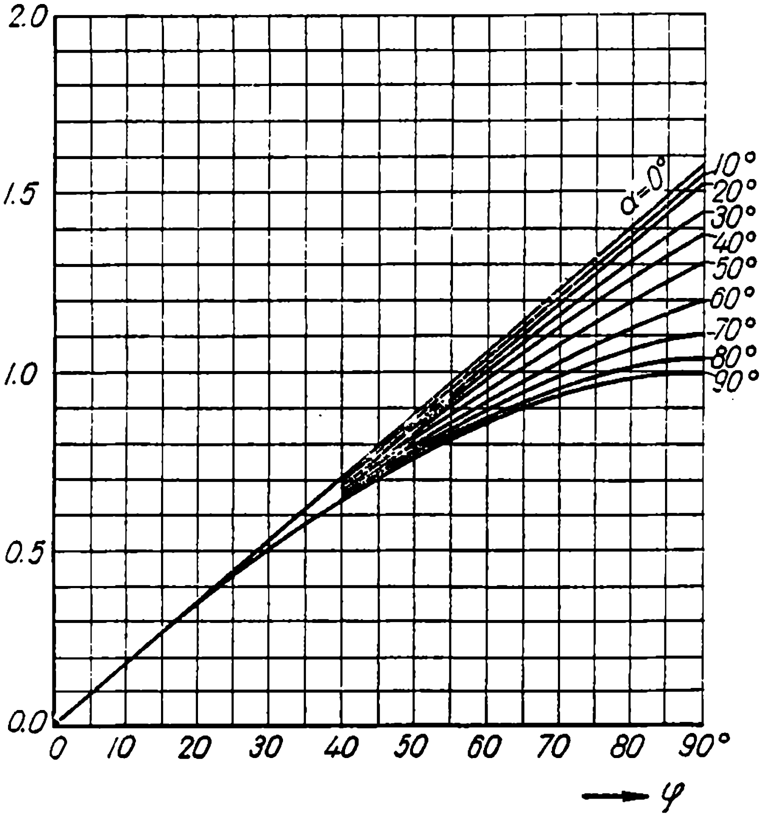


FIG. 6. Elliptic integral of the second kind $E(k, \varphi)$; $k = \sin\alpha = \text{const.}$

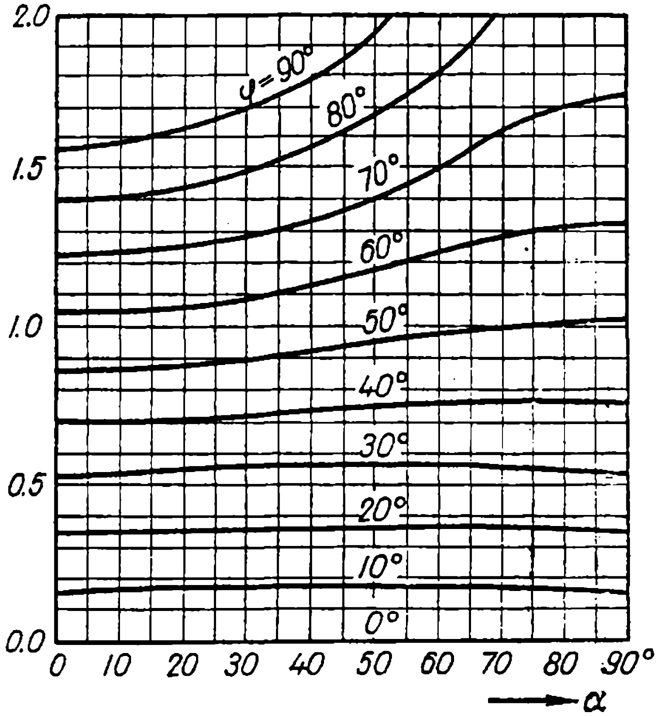


FIG. 7. Elliptic integral of the first kind $F(k, \varphi)$; $\varphi=$ const, $k=\sin\alpha$.

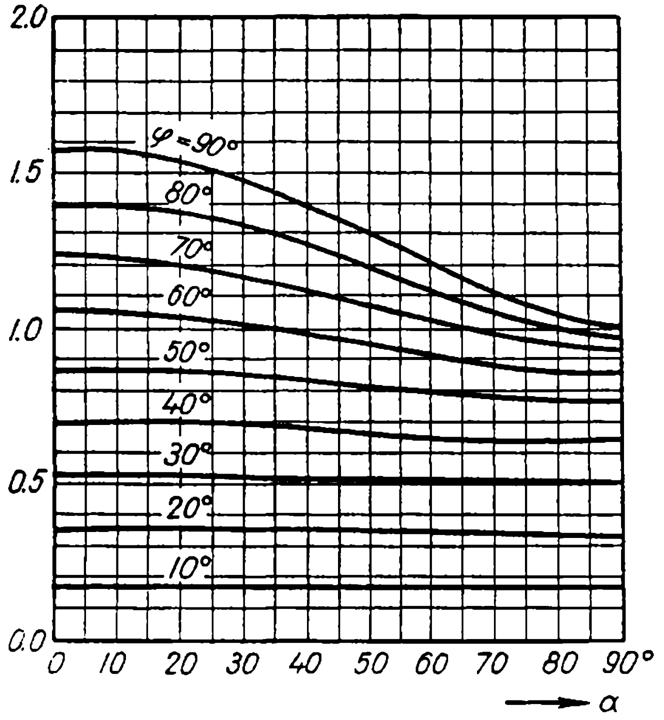


FIG. 8. Elliptic integral of the second kind $E(k, \varphi)$; $\varphi=$const., $k=\sin\alpha$.

The following formulae are employed when k is close to 1:

$$\begin{aligned}\boldsymbol{K} = m+\left(\frac{1}{2}\right)^2(m-1)k'^2+\left(\frac{1.3}{2.4}\right)^2\left(m-1-\frac{2}{3.4}\right)k'^4+\\ +\left(\frac{1.3.5}{2.4.6}\right)^2\left(m-1-\frac{2}{3.4}-\frac{2}{5.6}\right)k'^6+\ldots\end{aligned}\tag{6.387}$$

$$\begin{aligned}\boldsymbol{E} = 1+\frac{1}{2}\left(m-\frac{1}{1.2}\right)k'^2+\frac{1^2.3}{2^2.4}\left(m-\frac{2}{1.2}-\frac{1}{3.4}\right)k'^4+\\ +\frac{1^2.3^2.4}{2^2.4^2.6}\left(m-\frac{2}{1.2}-\frac{2}{3.4}-\frac{1}{5.6}\right)k'^6+\ldots.\end{aligned}\tag{6.388}$$

Here, $m = \ln 4/k'$.

Some integrals are:

$$\int\frac{\boldsymbol{K}\,dk}{k^2} = -\frac{1}{k}\boldsymbol{E} = \frac{k^2\boldsymbol{D}-\boldsymbol{K}}{k};\tag{6.389}$$

$$\int\frac{\boldsymbol{E}\,dk}{k^2} = \frac{k'^2\,\boldsymbol{K}-2\boldsymbol{E}}{k};\tag{6.390}$$

$$\int\boldsymbol{K}\,k\,dk = \boldsymbol{E}-k'^2\boldsymbol{K} = k^2(\boldsymbol{K}-\boldsymbol{D});\tag{6.391}$$

$$\int\boldsymbol{E}k\,dk = \frac{1+k^2}{3}\boldsymbol{E}-\frac{k'^2}{3}\boldsymbol{K}.\tag{6.392}$$

Graphs of the elliptic integrals are shown in Fig. 5, 6, 7 and 8.

2. Integral functions

DEFINITION. *The following definite integrals are known as integral functions:*

The *integral exponential function*

$$\mathrm{Ei}(x) = \int_{-\infty}^{x}\frac{e^t}{t}\,dt \qquad (x < 0).\tag{6.393}$$

The *integral logarithm*

$$\mathrm{li}(x) = \int_0^x\frac{dt}{\ln t} \qquad (0 < x < 1).\tag{6.394}$$

The *integral sine*

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,dt \qquad \left(\mathrm{si}\,0 = \frac{\pi}{2}\right). \tag{6.395}$$

The *integral cosine*

$$\mathrm{ci}\,(x) = -\int_x^\infty \frac{\cos t}{t}\,dt\,(x < 0). \tag{6.396}$$

In addition, the following notations are used:

$$\mathrm{Si}(x) = \frac{\pi}{2}+\mathrm{si}(x) = \int_0^x \frac{\sin t}{t}\,dt\left[\mathrm{Si}(\infty) = \frac{\pi}{2}\right]. \tag{6.397}$$

$$\mathrm{Ci}\,(x) = \mathrm{ci}\,(x). \tag{6.398}$$

When $x > 1$, the function $\mathrm{li}(x)$ is defined as

$$\lim_{e\to 0}\left[\int_0^{1-e}\frac{dt}{\ln t}+\int_{1+e}^{x}\frac{dt}{\ln t}\right].$$

When $x > 0$, the function $\mathrm{Ei}(x)$ takes complex values; the real part of $\mathrm{Ei}(x)$ is denoted by $\overline{\mathrm{Ei}}(x)$. The bar above the Ei is occasionally omitted.

Relationships between the integral functions are

$$\mathrm{Ei}\,(\ln x) = \mathrm{li}\,(x) \qquad (x < 1); \tag{6.399}$$

$$\overline{\mathrm{Ei}}\,(\ln x) = \mathrm{li}\,(x) \qquad (x > 1); \tag{6.400}$$

$$\mathrm{li}\,(e^x) = \mathrm{Ei}\,(x) \qquad (x < 0); \tag{6.401}$$

$$\mathrm{Ei}\,(ix) = \mathrm{ci}\,(x)+i\,\mathrm{si}\,(x); \tag{6.402}$$

$$\mathrm{Ei}\,(x \pm i0) = \overline{\mathrm{Ei}}\,(x) \mp \pi i; \tag{6.403}$$

$$\mathrm{Si}\,(-x) = -\mathrm{Si}\,(x); \tag{6.404}$$

$$\mathrm{si}\,(-x) = -\mathrm{si}\,(x) -\pi; \tag{6.405}$$

$$\mathrm{Ci}\,(-x) = \mathrm{Ci}\,(x) \pm \pi i \qquad (x > 0); \tag{6.406}$$

$$\mathrm{Si}\,(2x) = \frac{\sin^2 x}{x} + \int_0^x\left(\frac{\sin t}{t}\right)^2 dt. \tag{6.407}$$

Representations as series are:

$$\mathrm{Ei}\,(x) = \boldsymbol{C} + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{kk!} \qquad (x < 0); \tag{6.408}$$

$$\overline{\mathrm{Ei}}\,(x) = \boldsymbol{C} + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{kk!} \qquad (x > 0); \tag{6.409}$$

$$\mathrm{li}\,(x) = \boldsymbol{C} + \ln(-\ln x) + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{kk!} \qquad (0 < x < 1); \tag{6.410}$$

$$\mathrm{li}\,(x) = \boldsymbol{C} + \ln\ln x + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{kk!} \qquad (x > 1); \tag{6.411}$$

$$\mathrm{si}\,(x) = -\frac{\pi}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(2k-1)!}; \tag{6.412}$$

$$\mathrm{ci}\,(x) = \boldsymbol{C} + \ln x + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!}. \tag{6.413}$$

Here, $\boldsymbol{C} = 0{\cdot}57721\ 56649\ldots$ is Euler's constant (see p. 310).

Approximate formulae (for small values of x) are:

$$\mathrm{li}\,(x) \approx -\frac{x}{\ln \frac{1}{x}}; \tag{6.414}$$

$$\mathrm{Si}\,(x) \approx x; \tag{6.415}$$

$$\mathrm{si}\,(x) \approx x - \frac{\pi}{2}; \tag{6.416}$$

$$\mathrm{Ci}\,(x) \approx \overline{\mathrm{Ei}}\,(x) \approx \mathrm{Ei}\,(-x) \approx \boldsymbol{C} + \ln x. \tag{6.417}$$

Asymptotic formulae (for large values of x) are:

$$\overline{\mathrm{Ei}}\,(x) = \frac{e^x}{x}\left(1 + \frac{1!}{x} + \frac{2!}{x^2} + \frac{3!}{x^3} + \ldots\right); \tag{6.418}$$

$$\mathrm{si}\,(x) = -\frac{\cos x}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \ldots\right) - \\ -\frac{\sin x}{x}\left(\frac{1!}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \ldots\right) \approx -\frac{\cos x}{x}; \tag{6.419}$$

$$\operatorname{ci}(x) = \frac{\sin x}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x}-\dots\right)-$$
$$-\frac{\cos x}{x}\left(\frac{1!}{x}-\frac{3!}{x^3}+\frac{5!}{x^5}-\dots\right) \approx \frac{\sin x}{x}; \qquad (6.420)$$

$$\overline{\mathrm{Ei}}\,(x) \approx \frac{e^x}{x}; \qquad (6.421)$$

$$\mathrm{Ei}\,(-x) \approx \frac{e^{-x}}{-x}. \qquad (6.422)$$

Some numerical values are:

$$\mathrm{Ei}\,(-1) = -0{\cdot}21938\,3934\dots . \qquad (6.423)$$

$$\overline{\mathrm{Ei}}\,(1) = 1{\cdot}89511\,7816\dots . \qquad (6.424)$$

$$\mathrm{li}\,(1{\cdot}45136\,92346\dots) = 0. \qquad (6.425)$$

Some limits are:

$$\lim_{x\to\infty} \operatorname{si}(x) = -\pi; \qquad (6.426)$$

$$\lim_{x\to\infty} \operatorname{ci}(x) = \pm\pi i; \qquad (6.427)$$

$$\lim_{x\to+\infty} [x^{\varrho}\operatorname{si}(x)] = 0 \quad (\varrho < 1); \qquad (6.428)$$

$$\lim_{x\to+\infty} [x^{\varrho}\operatorname{ci}(x)] = 0 \quad (\varrho < 1); \qquad (6.429)$$

$$\lim_{x\to+\infty} e^{-x}\,\overline{\mathrm{Ei}}\,(x) = 0; \qquad (6.430)$$

$$\lim_{x\to+\infty} xe^{-x}\,\overline{\mathrm{Ei}}\,(x) = 1. \qquad (6.431)$$

Some integrals are:

$$\int_0^x \mathrm{Ei}\,(-mt)\,dt = x\,\mathrm{Ei}\,(-mx)-\frac{1-e^{-mx}}{m}; \qquad (6.432)$$

$$\int_0^\infty e^{-pt}\,\mathrm{Ci}\,(qt)\,dt = -\frac{1}{p}\ln\left(1+\frac{p^2}{q^2}\right); \qquad (6.433)$$

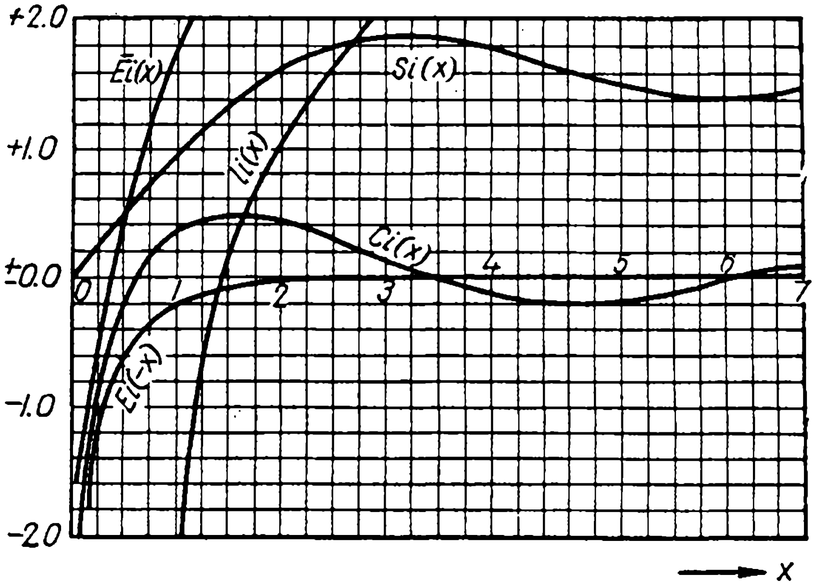


FIG. 9. Integral functions: the sine Si(x), cosine Ci(x), logarithm li (x) and exponential functions Ei(—x) and $\overline{\mathrm{Ei}}(x)$.

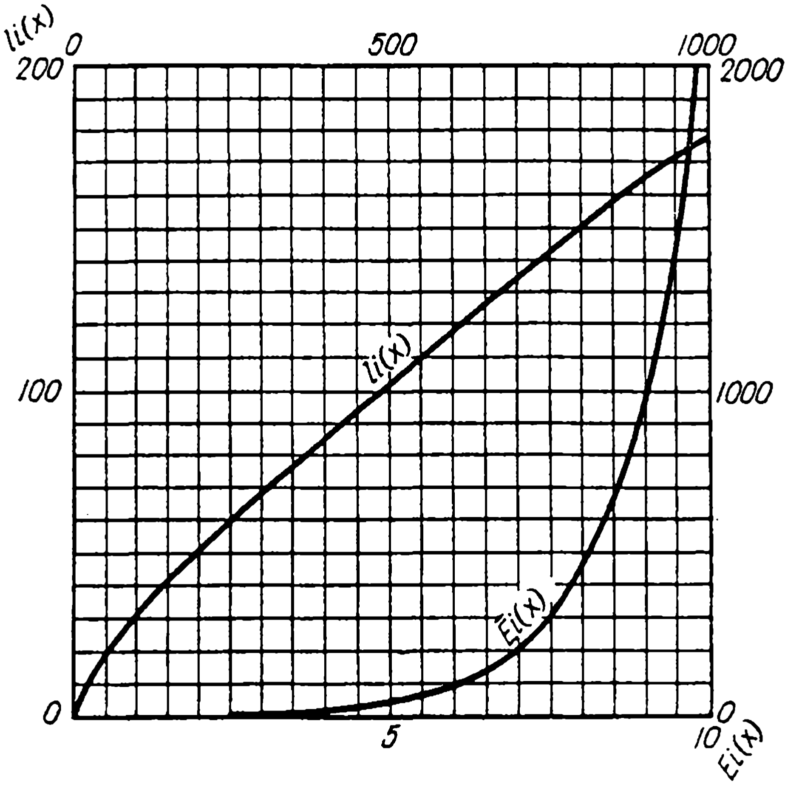


FIG. 10. The integral exponential function $\overline{\mathrm{Ei}}(x)$ and the integral logarithm li(x).

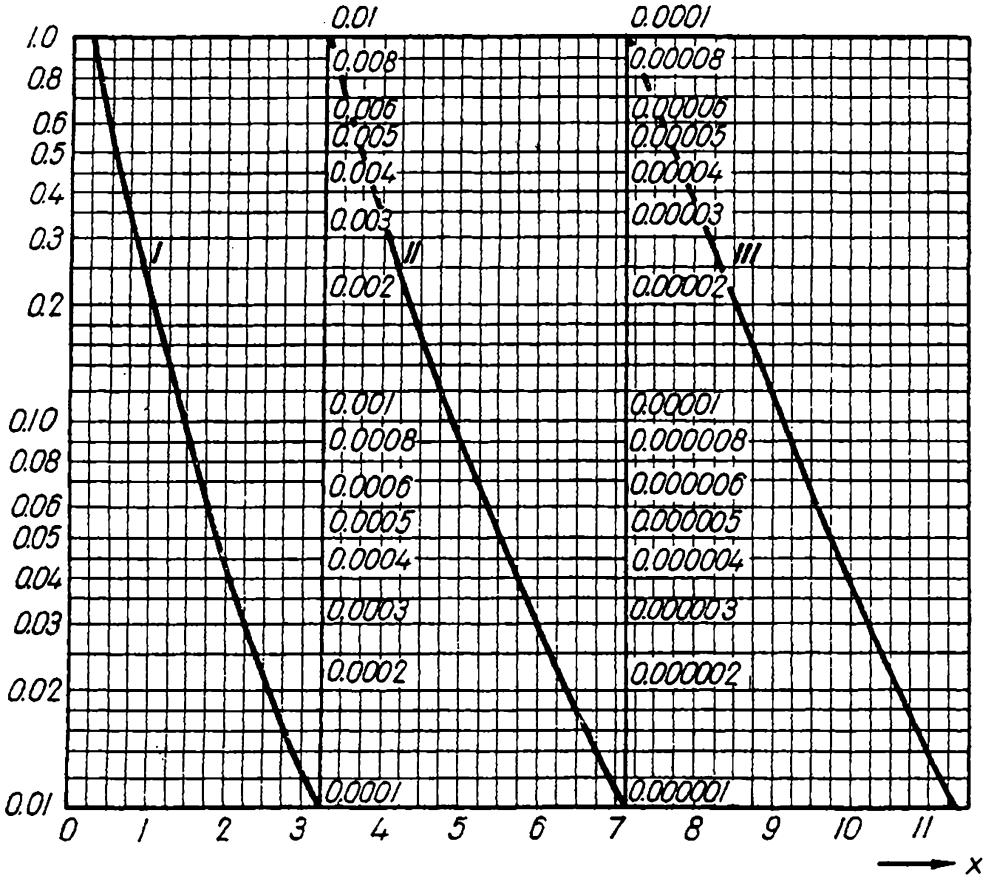


FIG. 11. The integral exponential function —Ei(x).

$$\int_0^\infty e^{-pt}\,\mathrm{si}\,(qt)\,dt = -\frac{1}{q}\arctan\frac{p}{q}; \tag{6.434}$$

$$\int_0^\infty \cos t\,\mathrm{Ci}\,(t)\,dt = \int_0^\infty \sin t\,\mathrm{si}\,(t)\,dt = -\frac{\pi}{4}; \tag{6.435}$$

$$\int_0^\infty \mathrm{ci}^2\,(t)\,dt = \int_0^\infty \mathrm{si}^2\,(t)\,dt = \frac{\pi}{2}; \tag{6.436}$$

$$\int_0^\infty \mathrm{Ci}\,(t)\,\mathrm{si}\,(t)\,dt = -\ln 2; \tag{6.437}$$

$$\int_0^\infty \mathrm{ci}\,(\alpha t)\,\mathrm{ci}\,(\beta t)\,dt = \int_0^\infty \mathrm{si}\,(\alpha t)\,\mathrm{si}\,(\beta t)\,dt = \begin{cases} \dfrac{\pi}{2\alpha} & \text{for } \alpha > \beta, \\ \dfrac{\pi}{2\beta} & \text{for } \alpha < \beta. \end{cases} \tag{6.438}$$

The graphs of the integral functions are shown in Fig. 9, 10 and 11.

3. The error function

DEFINITION. *The error function is defined as the function*

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \qquad [\operatorname{erf}(\infty) = 1]. \tag{6.439}$$

It is often defined as

$$\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{t^2}{2}}\,dt \tag{6.440}$$

or (by S. N. Bernshtein) as

$$\Phi_{\text{B}}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}}\,dt = \frac{1}{2}\Phi(x) = \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right). \tag{6.441}$$

The following notations are also encountered:

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt, \tag{6.442}$$

$$L(x) = \int_x^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erfc} x. \tag{6.443}$$

Relationships between the functions erf x and $\varphi(x)$ are:

$$\Phi(x) = \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right); \tag{6.444}$$

$$\operatorname{erf} x = \Phi(x\sqrt{2}). \tag{6.445}$$

The derivatives of the error function are:

$$\frac{d}{dx}(\operatorname{erf} x) = \frac{2}{\sqrt{\pi}} e^{-x^2}; \tag{6.446}$$

$$\frac{d}{dx}\Phi(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}. \tag{6.447}$$

The integral of the error function is:

$$\int \operatorname{erf} x\,dx = x \operatorname{erf} x + \frac{1}{\sqrt{\pi}} e^{-x^2} + C. \tag{6.448}$$

Other integral forms are:

$$\operatorname{erf} x = \frac{1}{\sqrt{\pi}} \int_0^{x^2} \frac{e^{-t}}{\sqrt{t}}\, dt; \tag{6.449}$$

$$\operatorname{erf}(xy) = \frac{2y}{\sqrt{\pi}} \int_0^{x} e^{-y^2 t^2}\, dt; \tag{6.450}$$

$$\operatorname{erf}(\sqrt{qx}) = \sqrt{\frac{q}{\pi}} \int_0^{x} \frac{e^{-qt}}{\sqrt{qt}}\, dt. \tag{6.451}$$

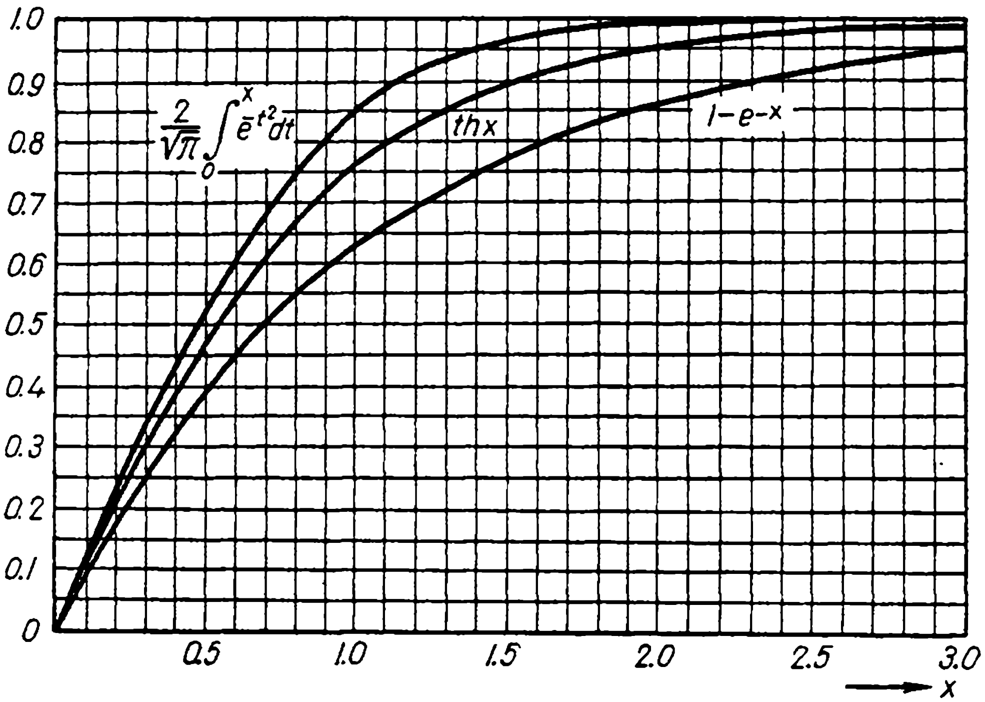


FIG. 12. The error function $\operatorname{erf} x = (2/\sqrt{\pi}) \int_0^x e^{-t^2}\, dt$.

The representation as a series is

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!} = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!!}. \tag{6.452}$$

Asymptotic formulae are:

$$\operatorname{erf}(\sqrt{x}) = 1 - \frac{1}{\pi} e^{-x} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(k+\frac{1}{2}\right)}{x^{k+\frac{1}{2}}} + \frac{e^{-x}}{\pi} \mathrm{R}_n, \tag{6.453}$$

where

$$|R| < \frac{\Gamma\left(n+\frac{1}{2}\right)}{|x|^{n+\frac{1}{2}}\cos\frac{\varphi}{2}}, \quad x = xe^{i\varphi} \text{ and } \varphi^2 < \pi^2; \tag{6.454}$$

$$\operatorname{erfc} x \approx \frac{e^{-x^2}}{\sqrt{\pi}x}\left[1 - \frac{1}{2x^2} + \frac{1.3}{(2x^2)^2} - \frac{1.3.5}{(2x^2)^3} + \dots\right]. \tag{6.455}$$

The graph of the function erf x is shown is Fig. 12, which also gives the graphs of tanh x and $1 - e^x$ for comparison.

4. Fresnel integrals

Definition. *The following functions are called Fresnel integrals:* the *Fresnel sine integral,*

$$S(x) = \sqrt{\frac{2}{\pi}}\int_0^x \sin t^2\,dt \quad \left[S(+\infty) = \frac{1}{2}\right], \tag{6.456}$$

the *Fresnel cosine integral,*

$$C(x) = \sqrt{\frac{2}{\pi}}\int_0^x \cos t^2\,dt \left[C(+\infty) = \frac{1}{2}\right]. \tag{6.457}$$

The following definitions of the Fresnel integrals are also occasionally encountered:

$$S^*(x) = \int_0^x \sin\frac{\pi}{2}t^2\,dt = S\left(\sqrt{\frac{\pi}{2}}x\right), \tag{6.458}$$

$$C^*(x) = \int_0^x \cos\frac{\pi}{2}t^2\,dt = C\left(\sqrt{\frac{\pi}{2}}x\right). \tag{6.459}$$

Other integral forms are:

$$S(x) = \frac{1}{\sqrt{2\pi}}\int_0^{x^2}\frac{\sin t}{\sqrt{t}}\,dt; \tag{6.460}$$

$$C(x) = \frac{1}{\sqrt{2\pi}}\int_0^{x^2}\frac{\cos t}{\sqrt{t}}\,dt, \tag{6.461}$$

$$S(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \sin(y^2 t^2)\, dt; \tag{6.462}$$

$$C(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \cos(y^2 t^2)\, dt. \tag{6.463}$$

The representations as series are:

$$S(x) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+3}}{(2k+1)!\,(4k+3)}; \tag{6.464}$$

$$C(x) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{(2k)!\,(4k+1)}. \tag{6.465}$$

The asymptotic formulae are:

$$S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi x}} \cos x^2 + O\left(\frac{1}{x^2}\right) \quad (x \to \infty); \tag{6.466}$$

$$C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi x}} \sin x^2 + O\left(\frac{1}{x^2}\right) \quad (x \to \infty). \tag{6.467}$$

The connections between the Fresnel integrals, Bessel functions and the error function are:

$$S(x) = \frac{1}{2} \int_0^{x^2} J \frac{1}{2}(t)\, dt; \tag{6.468}$$

$$C(x) = \frac{1}{2} \int_0^{x^2} J \frac{1}{2}(t)\, dt; \tag{6.469}$$

$$C(z) + iS(z) = \sqrt{\frac{i}{2}} \operatorname{erf}\left(\frac{z}{\sqrt{i}}\right) = \sqrt{\frac{2}{\pi}} \int_0^z e^{it^2}\, dt; \tag{6.470}$$

$$C(z) - iS(z) = \frac{1}{\sqrt{2i}} \operatorname{erf}(z\sqrt{i}) = \sqrt{\frac{2}{\pi}} \int_0^z e^{-it^2} dt. \tag{6.471}$$

Some limits are:

$$\lim_{x \to +\infty} S(x) = \lim_{x \to +\infty} C(x) = \frac{1}{2}; \tag{6.472}$$

$$\lim_{x \to +\infty} \left\{ x^{\varrho} \left[S(x) - \frac{1}{2} \right] \right\} = \lim_{x \to +\infty} \left\{ x^{\varrho} \left[C(x) - \frac{1}{2} \right] \right\} = 0 \quad (\varrho < 1). \tag{6.473}$$

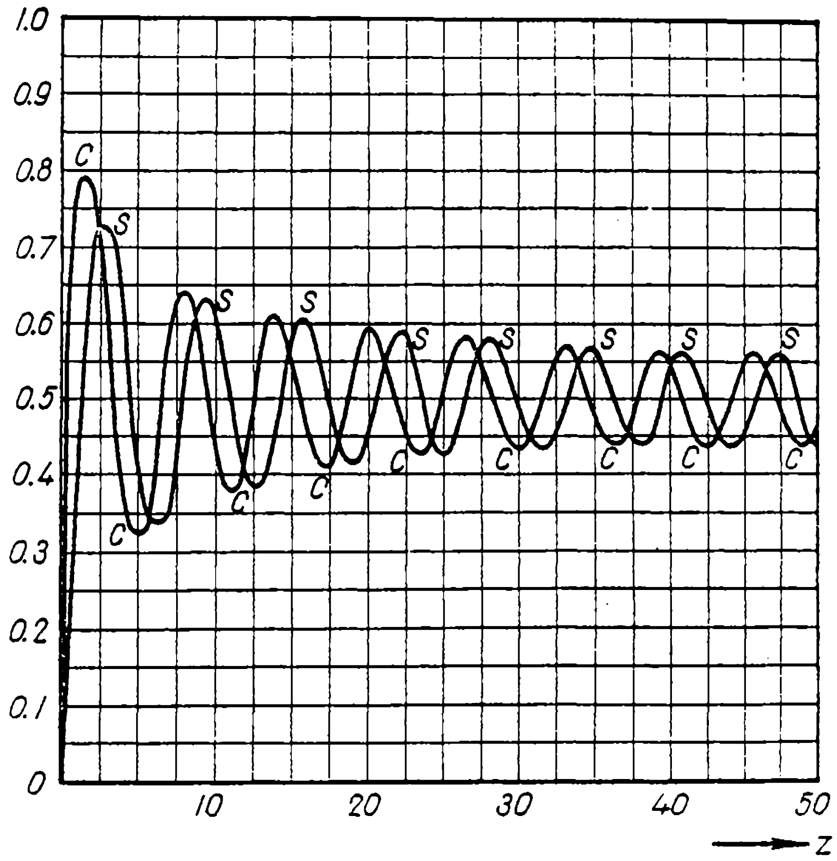


FIG. 13. The Fresnel integrals $S(x)$ and $C(x)$, $x^2=z$.

Some integrals are:

$$\int_0^p S(\alpha x)\,dx = pS(\alpha p)+\frac{\cos(\alpha^2p^2)-1}{\alpha\sqrt{2\pi}}; \tag{6.474}$$

$$\int_0^p C(\alpha x)\,dx = pC(\alpha p)-\frac{\sin(\alpha^2p^2)}{\alpha\sqrt{2\pi}}; \tag{6.475}$$

$$\int_0^\infty \left[\frac{1}{2}-S(x)\right]\sin 2px\,dx = -\frac{2\sqrt{2}\cos\frac{\pi}{8}}{\pi}\,\frac{\sin\frac{p^2}{2}}{p} \quad (p>0); \tag{6.476}$$

$$\int_0^\infty \left[\frac{1}{2}-C(x)\right]\sin 2px\,dx = -\frac{2\sqrt{2}\sin\frac{\pi}{8}}{\pi}\,\frac{\sin\frac{p^2}{2}}{p} \quad (p>0). \tag{6.477}$$

The graphs of the Fresnel integrals are shown in Fig. 13.

5. Gamma and beta functions of Euler

The so-called *Euler B*- and *Γ-functions* are important non-elementary functions. They are defined by improper integrals dependent on a parameter; they have been carefully studied, and fairly detailed tables have been compiled for them.

The fundamental properties of the functions are described below. All the parameters appearing in the formulae are real. The operations employed, of differentiation, integration, change of the order of integrations, summation, etc., are valid, inasmuch as the integrals are absolutely and uniformly convergent for the values of the parameters considered.

1°. *Definitions, functional equations and elementary properties of the Γ (gamma)-function of Euler.* For $\alpha > 0$, Euler's gamma function is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx. \tag{6.478}$$

Change of the variables leads to the formulae:

$$\Gamma(\alpha) = \int_0^1 \left(\ln \frac{1}{x}\right)^{\alpha-1} dx; \tag{6.479}$$

$$\Gamma(\alpha) = \int_{-\infty}^{\infty} e^{\alpha x} e^{-e^{x}}\, dx. \tag{6.480}$$

Integration by parts of (6.478) gives

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1) \tag{6.481}$$

and in general, for integral n and $\alpha > 0$, we have

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)\ldots(\alpha-n+1)\,\Gamma(\alpha-n+1). \tag{6.482}$$

On putting $\alpha = n$ (integer) in (6.482), we get

$$\Gamma(n) = (n-1)! \tag{6.483}$$

Continuation of the gamma function into the negative semi-axis is accomplished in accordance with the formula

$$\Gamma(\alpha-1) = \frac{\Gamma(\alpha)}{\alpha-1} \tag{6.484}$$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \tag{6.485}$$

first into the interval $(-1, 0)$, then into $(-2, -1)$, so that, for $-n < x < -(n-1)$,

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)\dots(x+n-1)}, \tag{6.486}$$

whence, for $0 < \alpha < 1$,

$$\Gamma(\alpha-n) = (-1)^n \frac{\Gamma(\alpha)}{(1-\alpha)(2-\alpha)\dots(n-\alpha)}. \tag{6.487}$$

The function $\Gamma(x+1)$ is often denoted by the symbol $\Pi(x)$.

The gamma function is discontinuous for all integral negative values of the argument, since it follows from (6.485) that

$$\lim_{\substack{x \to -n \\ x > -n}} |x^{-1}\Gamma(x+1)| = \infty. \tag{6.488}$$

It follows from (6.478) that, at points where the gamma function is continuous, its derivatives may be evaluated from the formulae

$$\left.\begin{aligned} \Gamma'(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}\ln x\, dx, \\ \Gamma^{(k)}(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}(\ln x)^k dx. \end{aligned}\right\} \tag{6.489}$$

Remark. An equation of the form

$$F(x,\ y(x),\ y'(x),\ \dots\ y^{(n)}(x)) = 0$$

where F is a polynomial in its arguments x, $y(x)$, $y'(x)$, ..., $y^{(n)}(x)$, is called an *algebraic differential equation* of the nth order. A function $y(x)$ is described as *hypertranscendental* if it is not a solution of any algebraic differential equation with polynomial coefficients. The function $\Gamma(x)$ is hypertranscendental.

For instance, $y(x) = \sin x$ is a solution of the algebraic equation $y'^2+y^2 = 1$, so that it is not a hypertrascendental function.

An elementary asymptotic formula. It follows from the inequality with $0 < \alpha < 1$)

$$n^\alpha\Gamma(n) - e^{-n}n^{\alpha+n-1} < \Gamma(\alpha+n) =$$

$$= \int_0^n + \int_n^\infty < n^{\alpha-1}\Gamma(n+1) + e^{-n}n^{\alpha+n-1} \tag{6.490}$$

that

$$\lim_{n\to\infty} \frac{\Gamma(\alpha+n)}{n!n^{\alpha-1}} = 1. \tag{6.491}$$

The representation of the gamma function as a series is mentioned in Chapter III, § 3. sec. 16.

Expressions for $\Gamma(\alpha)$ and $\Gamma^{-1}(\alpha)$ as infinite products. If we use (6.491) and replace $\Gamma(\alpha+n)$ in accordance with (6.482), we arrive at the *Euler–Gauss formula*

$$\Gamma(\alpha) = \lim_{n\to\infty} n^{\alpha} \frac{n!}{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n)}, \tag{6.492}$$

or

$$\Gamma(\alpha) = \lim_{n\to\infty} n^{\alpha} \frac{n!}{\prod\limits_{\nu=0}^{n} (\alpha+\nu)}. \tag{6.493}$$

We can obtain from these last formulae, on making use of the transformation

$$\frac{\prod\limits_{\nu=0}^{n} (\alpha+\nu)}{n!n^{\alpha}} = \frac{\alpha}{n^{\alpha}} \cdot \prod_{\nu=1}^{n}\left(1+\frac{\alpha}{\nu}\right) =$$

$$= e^{-\alpha \ln n + \sum\limits_{\nu=1}^{n} \frac{\alpha}{\nu}} \alpha \prod_{\nu=1}^{n}\left(\left(1+\frac{\alpha}{\nu}\right)e^{-\frac{\alpha}{\nu}}\right),$$

the *Weierstrass formula*

$$\frac{1}{\Gamma(\alpha)} = \Gamma^{-1}(\alpha) = e^{C\alpha}\alpha \prod_{\nu=1}^{\infty}\left(\left(1+\frac{\alpha}{\nu}\right)e^{-\frac{\alpha}{\nu}}\right), \tag{6.494}$$

where $\mathbf{C}$ is the Euler–Mascheroni constant (6.56).

The formula for the complement

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} \tag{6.495}$$

follows from (6.493) and the formula

$$\frac{\sin \pi\alpha}{\pi\alpha} = \lim_{n\to\infty} \prod_{\nu=1}^{n}\left(1-\frac{\alpha^2}{\nu^2}\right).$$

In particular, it follows from (6.495) that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{6.496}$$

Since we have, from the formula for the complement,

$$\left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\ldots\left(\frac{n-1}{n}\right)\right]^2 = \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi}, \tag{6.497}$$

we obtain, in view of the relationship

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$$

the *Euler product*:

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}. \tag{6.498}$$

The multiplication formula

$$\prod_{\nu=0}^{k-1}\Gamma\left(\alpha+\frac{\nu}{k}\right) = (2\pi)^{\frac{k-1}{2}}k^{-k\alpha+\frac{1}{2}}\Gamma(k\alpha),\ \alpha>0,\ k \text{ is an integer} \geqq 1, \tag{6.499}$$

follows from (6.493). When $k = 2$, we get the (Legendre) *doubling formula*

$$\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right) = \sqrt{2\pi}\,2^{-2\alpha+\frac{1}{2}}\Gamma(2\alpha). \tag{6.500}$$

The logarithm of the gamma function. On taking logarithms of (6.493) and passing to the integrals, we arrive at the integral form of the logarithm of the gamma function:

$$\ln\alpha\Gamma(\alpha) = \sum_{n=1}^{\infty}\left(\ln\frac{n}{n+\alpha}-\alpha\ln\frac{n}{n+1}\right) =$$

$$= \sum_{n=1}^{\infty}\int_{-\infty}^{0}\left(\frac{e^{nx}-e^{(n+\alpha)x}}{x}-\alpha\,\frac{e^{nx}-e^{(n+1)x}}{x}\right)dx =$$

$$= \int_{-\infty}^{0}\frac{1-e^{\alpha x}-\alpha(1-e^{x})}{x(1-e^{x})}\,e^{x}\,dx,$$

$$\ln\Gamma(\alpha) = \int_{-\infty}^{0}\left[\frac{e^{\alpha x}-e^{x}}{e^{x}-1}-(\alpha-1)e^{x}\right]\frac{dx}{x}. \tag{6.501}$$

Other integral forms are:

$$\ln \Gamma(\alpha) = \int_0^\infty \left[(\alpha-1)e^{-t} + \frac{(1+t)^{-\alpha}-(1+t)^{-1}}{\ln(1+t)} \right] \frac{dt}{t}, \quad \alpha > 0, \qquad (6.502)$$

$$\ln \Gamma(\alpha) = \int_0^1 \left[\frac{t^\alpha - t}{t-1} - t\,(\alpha-1) \right] \frac{dt}{t \ln t}, \qquad \alpha > 0. \qquad (6.503)$$

Raabe's formula. On taking logarithms of the Euler product (6.498), we get

$$\sum_{k=1}^{n} \ln \Gamma\left(\frac{k}{n}\right)\frac{1}{n} = \left(\frac{1}{2} - \frac{1}{2n}\right) \ln 2\pi - \frac{1}{2n} \ln n,$$

and if

$$\frac{k}{\alpha} = \alpha \, \frac{1}{n} = d\alpha, \qquad n \to \infty$$

we have

$$\int_0^1 \ln \Gamma(\alpha)\, d\alpha = \ln \sqrt{2\pi}. \qquad (6.504)$$

The *Raabe integral* is obtained by differentiation with respect to the parameter:

$$J = \int_0^1 \ln \Gamma(\alpha + x)\, dx = \alpha\,(\ln \alpha - l + \ln \sqrt{2\pi}). \qquad (6.505)$$

Another expression for the Raabe integral is obtained by integration of (6.501):

$$J = \int_{-\infty}^{0} \frac{dx}{x} \left[\frac{e^{\alpha x}}{x} - \frac{e^x}{e^x - 1} - \left(\alpha - \frac{1}{2}\right) e^x \right]. \qquad (6.506)$$

It follows from (6.501), (6.506) that

$$\ln \Gamma(\alpha) - J + \frac{\ln \alpha}{2} = \int_{-\infty}^{0} f(x) e^{\alpha x}\, dx, \qquad (6.507)$$

where

$$f(x) = \frac{1}{x}\left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right).$$

On introducing the *Binet function*

$$\omega(\alpha) = \int_{-\infty}^{0} f(x)e^{\alpha x}\,dx, \qquad (6.508)$$

we get an asymptotic expression for the logarithm of the gamma-function:

$$\ln \Gamma(\alpha) = \ln \sqrt{2\pi}+\left(\alpha-\frac{1}{2}\right)\ln\alpha-\alpha+\omega(\alpha). \qquad (6.509)$$

It also follows from (6.501) and (6.506) that

$$J-\ln\Gamma\left(\alpha+\frac{1}{2}\right) = \int_{-\infty}^{0} F(x)e^{\alpha x}\,dx, \qquad (6.510)$$

where

$$F(x) = \frac{1}{x}\left(\frac{1}{x}-\frac{e^{\frac{x}{2}}}{e^{x}-1}\right).$$

From this we obtain an asymptotic formula for $\Gamma\left(\alpha+\frac{1}{2}\right)$:

$$\ln\Gamma\left(\alpha+\frac{1}{2}\right) = \alpha\,(\ln\alpha-1)+\ln\sqrt{2\pi}-\omega_1(\alpha), \qquad (6.511)$$

where

$$\omega_1(\alpha) = \int_{-\infty}^{0} F(x)e^{\alpha x}\,dx. \qquad (6.512)$$

The following relationship holds:

$$\omega_1(\alpha) = \omega(\alpha)-\omega(2\alpha); \qquad (6.513)$$

it is obtained by taking logarithms of the doubling formula (6.500) and substituting for $\ln\Gamma(\alpha)$ and $\ln\Gamma(2\alpha)$ in accordance with (6.509) and for $\ln\Gamma\left(\alpha+\frac{1}{2}\right)$ in accordance with (6.511).

The following formulae are due to Schaar:

$$\omega(\alpha) = \int_{-\infty}^{0} f(x)e^{\alpha x}\,dx = \sum_{1}^{n} \int_{-\infty}^{0} \frac{2e^{\alpha x}\,dx}{x^2+4k^2\pi^2} =$$

$$= \frac{1}{\pi}\int_{0}^{-\infty} \frac{\alpha\,dx}{\alpha^2+x^2} \ln(1-e^{2\pi x}). \tag{6.514}$$

$$\omega_1(\alpha) = \frac{1}{\pi}\int_{0}^{-\infty} \frac{\alpha\,dx}{\alpha^2+x^2} \ln(1-e^{2\pi x}) - \frac{1}{\pi}\int_{0}^{-\infty} \frac{2\alpha\,dx}{4\alpha^2+x^2} \ln(1-e^{2\pi x}) =$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} \frac{\alpha\,dx}{\alpha^2+x^2} \ln(1+e^{2\pi x}). \tag{6.515}$$

The Stirling and Gauss asymptotic formulae. By using (6.179) and (6.508), we can obtain the *Stirling series* for the Binet function:

$$\omega(\alpha) = \frac{B_2}{1.2}\frac{1}{\alpha} + \frac{B_5}{3.4}\frac{1}{\alpha^3} + \dots$$

$$\dots + \frac{B_{2n-2}}{(2n-1)(2n-2)}\frac{1}{\alpha^{2n-3}} + R, \tag{6.516}$$

where

$$R = \theta\,\frac{B_{2n}}{(2n-1)2n}\frac{1}{\alpha^{2n-1}}, \qquad 0 < \theta < 1. \tag{6.517}$$

When $n = 1$,

$$\omega(\alpha) = \theta\,\frac{B_2}{2\alpha} = \frac{\theta}{12\alpha}$$

and *Stirling's formula* follows from (6.509):

$$\ln \Gamma(\alpha) = \ln\sqrt{2\pi} + \left(\alpha - \frac{1}{2}\right)\ln\alpha - \alpha + \frac{\theta}{12\alpha}, \tag{6.518}$$

$$\Gamma(\alpha) = \sqrt{2\pi}\,\alpha^{\alpha-\frac{1}{2}}\,e^{-\alpha+\frac{\theta}{12\alpha}}. \tag{6.519}$$

Stirling's formula is again obtained when α is equal to an integer m:

$$m! = \sqrt{2\pi m}\left(\frac{m}{e}\right)^{m} e^{\frac{\theta}{12m}}. \tag{6.520}$$

An asymptotic formula for $\omega_1(\alpha)$ follows from (6.513) and (6.516):

$$\omega_1(\alpha) = \frac{B_2}{1}\left(1-\frac{1}{2}\right)\frac{1}{\alpha}+\frac{B_4}{3.4}\left(1-\frac{1}{2^3}\right)\frac{1}{\alpha^3}+ \\ +\frac{B_6}{5.6}\left(1-\frac{1}{2^5}\right)\frac{1}{\alpha^5}+\dots. \qquad (6.521)$$

In particular,

$$\omega_1(\alpha) = \theta\frac{B_2}{1.2}\left(1-\frac{1}{2}\right)\frac{1}{\alpha} = \frac{\theta}{24\alpha}, \qquad 0 < \theta < 1. \qquad (6.522)$$

Gauss's formula follows from (6.522):

$$\ln \Gamma\left(\alpha+\frac{1}{2}\right) = \alpha(\ln\alpha - 1) + \ln\sqrt{2\pi} - \frac{\theta}{24\alpha}. \qquad (6.523)$$

When n is an integer and $\alpha = n+\frac{1}{2}$, a formula is obtained for a factorial that gives a better approximation than Stirling's formula:

$$n! = \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} e^{-\frac{\theta}{24n+12}}. \qquad (6.524)$$

The power expansion of the logarithm of a gamma function. It follows from (6.481) and (6.493) that

$$\log\Gamma(1+\alpha) =$$

$$= \int_1^\alpha \frac{d\alpha}{\alpha} + \int_1^\alpha \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\,d\alpha = \int_1^\alpha \left[-\mathbf{C}+\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu}-\frac{1}{\nu+\alpha}\right)\right]d\alpha =$$

$$= \int_1^\alpha\left[-\mathbf{C}+\sum_{\nu=1}^{\infty}\left(\frac{\alpha}{\nu^2}-\frac{\alpha^2}{\nu^3}+\frac{\alpha^3}{\nu^4}-\frac{\alpha^4}{\nu^5}+\dots\right)\right]d\alpha =$$

$$= -\mathbf{C}\alpha+\frac{s_2}{2}\alpha^2-\frac{s_3}{3}\alpha^3+\frac{s_4}{4}\alpha+\dots, \qquad (6.525)$$

where

$$s_p = 1+\frac{1}{2^p}+\dots+\frac{1}{n^p}+\dots.$$

The trigonometric expansion of the logarithm of the gamma function (Kummer's series). If, when $x \in (0, 1)$,

$$\ln \Gamma(\alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2n\pi\alpha + b_n \sin 2n\pi x), \tag{6.526}$$

where

$$\left.\begin{aligned} a_n &= 2\int_0^1 \ln \Gamma(x) \cos 2n\pi x \, dx \quad (n = 0,\ 1,\ 2,\ \ldots), \\ b_n &= 2\int_0^1 \ln \Gamma(x) \sin 2n\pi x \, dx \quad (n = 1,\ 2,\ 3,\ \ldots), \end{aligned}\right\} \tag{6.527}$$

we have by the formula for the complement:

$$\begin{aligned} \ln \Gamma(\alpha) + \ln \Gamma(1-\alpha) &= a_0 + \sum_{n=1}^{\infty} 2a_n \cos 2n\pi\alpha = \\ &= \ln 2\pi - \ln \sin \pi x = \ln 2\pi + \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{n}, \end{aligned} \tag{6.528}$$

whence

$$\frac{1}{2} a_0 = \ln \sqrt{2\pi}, \quad a_n = \frac{1}{2n} \qquad (n = 1,\ 2,\ \ldots). \tag{6.529}$$

We have from (6.502) and (6.64):

$$\begin{aligned} b_n &= 2\int_0^1 \frac{dt}{\ln t} \int_0^1 \left(\frac{1 - t^{x-1}}{1-t} x + 1 \right) \sin 2n\pi x \, dx = \\ &= \frac{1}{n\pi} \int_0^{\infty} \left(\frac{1}{1+u^2} - e^{-2n\pi u} \right) \frac{du}{u} = \frac{1}{n\pi} \int_0^{\infty} \left(\frac{1}{1+u^2} - e^{-u} \right) \frac{du}{u} + \\ &\quad + \frac{1}{n\pi} \int_0^{\infty} (e^{-u} - e^{-2n\pi u}) \frac{du}{u} = \frac{1}{n\pi} (\mathbf{C} + \ln 2n\pi). \end{aligned} \tag{6.530}$$

Further expansions of the logarithm of the gamma function are

$$\ln \Gamma(1+\alpha) = \frac{1}{2} \frac{\pi\alpha}{\sin \pi\alpha} - \mathbf{C}\alpha - \frac{s_3}{3}\alpha^3 - \frac{s_5}{5}\alpha^5 - \ldots, \tag{6.531}$$

$$\begin{aligned} \ln \Gamma(1+\alpha) &= \frac{1}{2} \frac{\pi\alpha}{\sin \pi\alpha} - \frac{1}{2} \ln \frac{1+\alpha}{1-\alpha} + \\ &\quad + (1-\mathbf{C})\alpha - (s_3 - 1)\frac{\alpha^3}{3} - (s_5 - 1)\frac{\alpha^5}{5} - \ldots. \end{aligned} \tag{6.532}$$

The logarithmic derivatives of the gamma function (the psi or digamma function). By definition,

$$\begin{aligned}\psi(\alpha) &= \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \frac{d}{d\alpha}\ln\Gamma(\alpha);\\ \Psi(\alpha) &= \frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)} = \frac{d}{d\alpha}\ln\Gamma(\alpha+1),\end{aligned} \tag{6.533}$$

so that

$$\Psi(\alpha) = \psi(\alpha)+\frac{1}{\alpha}.$$

The function $\psi(\alpha)$ satisfies the functional equations:

$$\psi(1+\alpha)-\psi(a) = \frac{1}{\alpha}. \tag{6.534}$$

$$\psi(\alpha)-\psi(1-\alpha) = -\pi\tan\pi\alpha. \tag{6.535}$$

$$\psi(\alpha)-\psi(-\alpha) = -\pi\tan\pi\alpha-\frac{1}{\alpha}. \tag{6.536}$$

$$\psi(1+\alpha)-\psi(1-\alpha) = -\pi\tan\pi\alpha+\frac{1}{\alpha}. \tag{6.537}$$

$$\psi\left(\frac{1}{2}+\alpha\right)-\psi\left(\frac{1}{2}-\alpha\right) = \pi\tan\pi\alpha. \tag{6.538}$$

$$\psi(m\alpha) = m^{-1}\sum_{r=0}^{m-1}\psi\left(\alpha+\frac{r}{m}\right)+\ln m. \tag{6.539}$$

It follows from (6.492) that

$$\psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = -\frac{1}{\alpha}-\mathbf{C}+\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu}-\frac{1}{\nu+\alpha}\right), \tag{6.540}$$

or, in view of the fact that

$$0 = 1-\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu}-\frac{1}{\nu+1}\right),$$

we have

$$\psi(\alpha) = -\mathbf{C}+\sum_{\nu=0}^{\infty}\left(\frac{1}{\nu+1}-\frac{1}{\nu+\alpha}\right). \tag{6.541}$$

It follows from (6.541) that

$$\psi(\alpha) = -\mathbf{C}+\sum_{\nu=0}^{\infty}\left(\int_0^1 t^\nu dt-\int_0^1 t^{\nu+\alpha-1}dt\right) =$$

$$= -\mathbf{C}+\int_0^1 \frac{1-t^{\alpha-1}}{1-t}\,dt. \qquad (6.542)$$

On carrying out the change of variable $1+y = t^{-1}$, and taking account of (6.63), *Cauchy's formula* is obtained:

$$\psi(\alpha) = \int_0^\infty \left[e^{-y}-\frac{1}{(1+y)^\alpha}\right]\frac{dy}{y}, \qquad \alpha > 0. \qquad (6.543)$$

Other integral forms are:

$$\psi(\alpha) = -\mathbf{C}+\int_0^\infty \frac{e^{-t}-e^{-t\alpha}}{1-e^{-t}}\,dt, \qquad \alpha > 0; \qquad (6.544)$$

$$\psi(\alpha) = \int_0^\infty \left(\frac{e^{-t}}{t}-\frac{e^{-t\alpha}}{1-e^{-t}}\right)dt, \qquad \alpha > 0; \qquad (6.545)$$

$$\psi(\alpha) = \ln\alpha-\frac{1}{2\alpha}-2\int_0^\infty \frac{t\,dt}{(t^2+\alpha^2)(e^{2\pi t}-1)}\,; \qquad (6.546)$$

$$\psi(\alpha) = \int_0^1 \left(\frac{1}{-\ln t}-\frac{t^{\alpha-1}}{1-t}\right)dt \qquad \alpha > 0; \qquad (6.547)$$

$$\psi(\alpha) = \int_0^\infty [(1+t)^{-1}-(1+t)^{-\alpha}]\frac{dt}{t}-\mathbf{C}, \qquad \alpha > 0. \qquad (6.548)$$

Some particular values of the gamma function and its derivatives are:

$$\Gamma(1) = \Gamma(2) = 1; \qquad (6.549)$$

$$\Gamma(0{\cdot}5) = \sqrt{\pi}\,; \qquad (6.550)$$

$$\Gamma(-0{\cdot}5) = -2\sqrt{\pi}\,; \qquad (6.551)$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!; \qquad (6.552)$$

$$\Gamma\left(\frac{1}{2}-n\right) = (-1)^n\frac{2^n\sqrt{\pi}}{(2n-1)!!}; \qquad (6.553)$$

$$\psi(1) = -\mathbf{C}; \qquad (6.554)$$

$$\psi\left(\frac{1}{2}\right) = -\mathbf{C}-2\ln 2; \tag{6.555}$$

$$\psi\left(\frac{1}{4}\right) = -\mathbf{C}-\frac{\pi}{2}-3\ln 2; \tag{6.556}$$

$$\psi\left(\frac{3}{4}\right) = -\mathbf{C}+\frac{\pi}{2}-3\ln 2; \tag{6.557}$$

$$\psi'(1) = \frac{\pi^2}{6}; \tag{6.558}$$

$$\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2}. \tag{6.559}$$

The graphs of $\Gamma(x)$, $\Pi(x)$, $1/\Gamma(x)$ and $1/\Pi(x)$ are shown in Fig. 14 and 15.

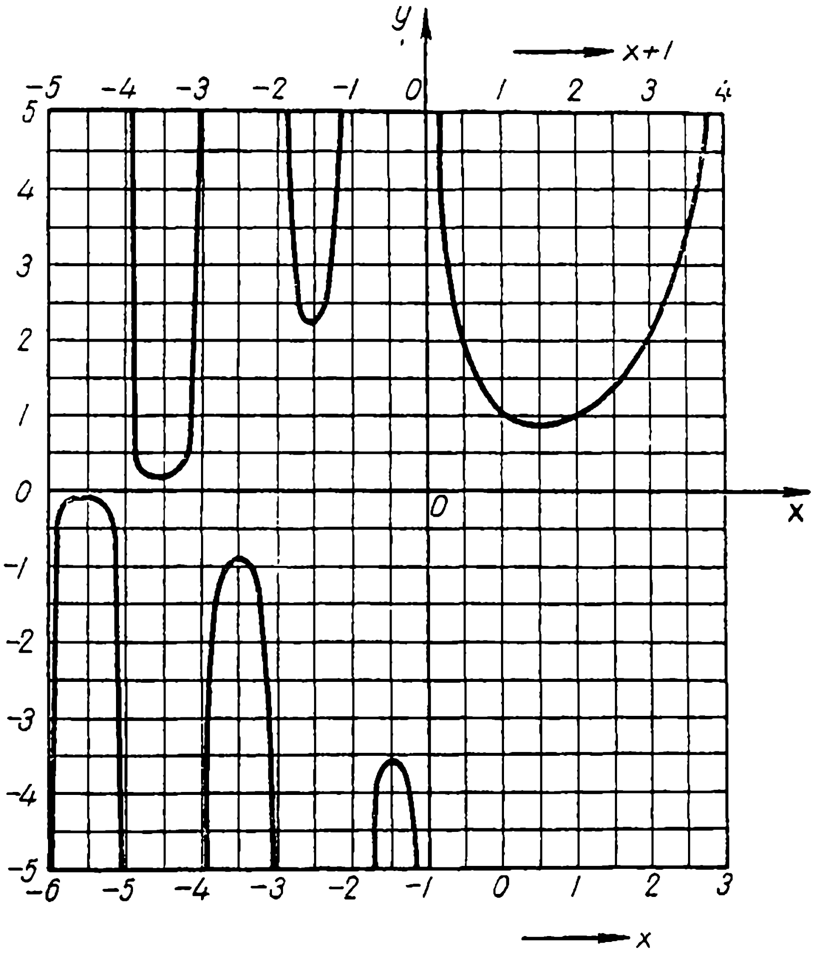


FIG. 14. The gamma functions $\Gamma(x)$ and $\Gamma(x+1) = \Pi(x)$.

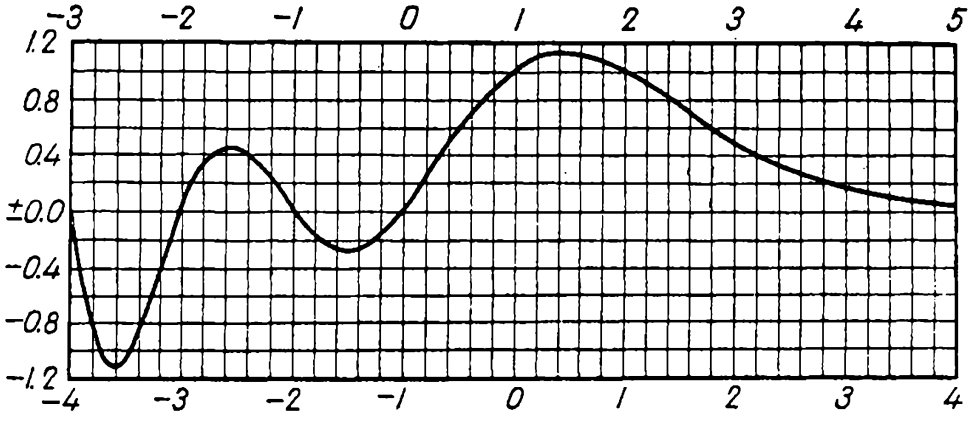


FIG. 15. The functions $1/\Gamma(x)$ and $1/\Pi(x)$.

Properties uniquely defining the gamma function. A function $F(x)$, continuous along with its derivative for $x > 0$, is a gamma function $\Gamma(x)$ if it satisfies any one of the groups of conditions enumerated below:

I. (a) $F(x+1) = xF(x)$,

(b) $F(x)F(1-x) = \dfrac{\pi}{\sin \pi x}$,

(c) $F(x)F\left(x+\dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2x-1}} F(2x)$.

II. (a) $F(x+1) = xF(x)$,

(b) $F(x)F\left(x+\dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2x-1}} F(2x)$,

(c) $F(x) \neq 0$ for $x > 0$.

III. (a) $F(1) = 1$,

(b) $F(x+1) = xF(x)$,

(c) $F(x)$ is a logarithmically convex function for $x > 0$ (see Chapter I, § 3, sec. 17).

IV. (a) $F(1) = 1$,

(b) $F(x+1) = xF(x)$,

(c) $(e/x)^x F(x)$ is decreasing for $x > 0$ (or $x > M$).

2°. *Definitions, functional equations and elementary properties of the B (beta)-function of Euler. Euler's beta function* is defined by the integral

$$B(\alpha,\ \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx, \qquad \alpha > 0, \qquad \beta > 0. \tag{6.560}$$

Integration by parts leads to the functional relationships

$$B(\alpha,\ \beta) = \frac{\beta-1}{\alpha+\beta-1}\,B(\alpha,\ \beta-1), \qquad \beta > 1, \tag{6.561}$$

$$B(\alpha,\ \beta) = \frac{\alpha-1}{\alpha+\beta-1}\,B(\alpha-1,\ \beta), \qquad \alpha > 1. \tag{6.562}$$

The B-function is symmetrical:

$$B(\alpha, \beta) = B(\beta, \alpha). \tag{6.563}$$

The connection between the binomial coefficients and their generalization. Since, when m and n are positive integers,

$$B(m,\ n) = \frac{(n-1)!(m-1)!}{(m+n-1)!}, \tag{6.564}$$

we have

$$\left.\begin{aligned} \frac{1}{nB(m, n)} &= \binom{m+n-1}{m-1}, \\ \frac{1}{mB(m, n)} &= \binom{m+n-1}{n-1}, \end{aligned}\right\} \tag{6.565}$$

whence, when $\alpha > 0$, $\beta > 0$,

$$\binom{a+b-1}{\beta-1} = \frac{1}{\alpha B(\alpha, \beta)}, \tag{6.566}$$

$$\binom{\alpha+\beta-1}{\beta-1} = \frac{1}{\beta B(\alpha, \beta)}. \tag{6.567}$$

An expression in terms of the gamma function. On replacing x in (6.560) by $y/(1+y)$, we arrive at the formula

$$B(\alpha, \beta) = \int_0^\infty \frac{y^{\alpha-1}dy}{(1+y)^{\alpha+\beta}}, \tag{6.568}$$

whence

$$B(\alpha, \beta)\Gamma(\alpha+\beta) = \int_0^\infty \frac{\Gamma(\alpha+\beta)y^{\alpha-1}\,dy}{(1+y)^{\alpha+\beta}} =$$

$$= \int_0^\infty\int_0^\infty e^{-x(1+y)}x^{\alpha+\beta-1}y^{\alpha-1}\,dx\,dy =$$

$$= \int_0^\infty e^{-x}x^{\beta-1}\,dx\int_0^\infty e^{-xy}(xy)^{\alpha-1}d(xy), \tag{6.569}$$

and finally,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{6.570}$$

Other integral forms are:

$$B(\alpha, \beta) = 2\int_0^{\pi/2} \sin^{2\alpha-1}\varphi\cos^{2\beta-1}\varphi d\varphi; \tag{6.571}$$

$$B(\alpha, \beta) = 2\int_0^{\pi/2} \sin^{2\alpha}\varphi\cos^{2\beta}\varphi d\varphi, \qquad \alpha > -\frac{1}{2}, \quad \beta > -\frac{1}{2}; \tag{6.572}$$

$$B(\alpha, \beta) = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt = 2\int_0^\infty \frac{t^{2\alpha-1}}{(1+t^2)^{\alpha+\beta}}\,dt, \tag{6.573}$$

$$\alpha > 0, \quad \beta > 0;$$

$$B(\alpha, \beta) = \int_0^1 \frac{t^{\alpha-1}+t^{\beta+1}}{(1+t)^{\alpha+\beta}}\,dt, \qquad \alpha > 0, \qquad \beta > 0; \tag{6.574}$$

$$B(\alpha, \alpha) = \frac{1}{2^{2\alpha-1}}\int_0^1 \frac{(1-t)^{\alpha-1}\,dt}{\sqrt{t}} \tag{6.575}$$

Representations as a series and infinite product:

$$B(\alpha, \beta) = \frac{1}{\beta}\sum_{n=0}^{\infty}(-1)^n\frac{\beta(\beta-1)\dots(\beta-n)}{n!(\alpha+n)}, \qquad \beta > 0; \tag{6.576}$$

$$B(\alpha, \beta) = \prod_{k=0}^{\infty}\frac{k(\alpha+\beta+k)}{(\alpha+k)(\beta+k)}\,. \tag{6.577}$$

6. Bessel functions

The Bessel functions of the first kind $J_n(x)$ are defined as the coefficients of the power expansion in the variable t of the generating function $e^{\frac{1}{2}\left(t-\frac{1}{t}\right)x}$:

$$e^{\frac{1}{2}(t-t^{-1})x} = e^{\frac{x}{2}t}e^{-\frac{x}{2}t^{-1}} =$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s x^{r+s}t^{r-s}}{2^{r+s}r!\,s!} = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \qquad (6.578)$$

$$J_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s x^{n+2s}}{2^{n+2s}(n+s)!\,s!}, \qquad -\infty < x < +\infty, \qquad (6.579)$$

$$J_{-n}(x) = (-1)^n J_n(x). \qquad (6.580$$

The Bessel functions of an imaginary argument $I_n(x)$ are defined from the relationship

$$e^{\frac{1}{2}(t+t^{-1})x} = \sum_{n=-\infty}^{\infty} I_n(x)t^n; \qquad (6.581)$$

$$I_n(x) = \sum_{s=0}^{\infty}\frac{x^{n+2s}}{2^{n+2s}(n+s)!\,s!}, \qquad (6.582)$$

$$I_n(x) = i^{-n}J_n(ix), \qquad (6.583)$$

$$I_{-n}(x) = i^n J_{-n}(ix) = i^n(-1)^n J_n(ix) = i^{-n}J_n(ix) = I_n(x). \qquad (6.584)$$

The trigonometric forms of the generating functions. It follows from (6.578) with $t = e^{i\varphi}$ that

$$e^{ix\sin\varphi} = J_0(x) + 2iJ_1(x)\sin\varphi +$$

$$+ 2J_2(x)\cos 2\varphi + 2iJ_3(x)\sin 3\varphi + 2J_4(x)\cos 4\varphi + \ldots, \qquad (6.585)$$

whence

$$\cos(x\sin\varphi) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos 2k\varphi, \qquad (6.586)$$

$$\sin(x\sin\varphi) = 2\sum_{k=0}^{\infty} J_{2k+1}(x)\sin(2k+1)\varphi, \qquad (6.587)$$

and similarly,

$$e^{ix\cos\varphi} = J_0(x)+2\sum_{s=1}^{\infty} i^s J_s(x)\cos s\varphi, \tag{6.588}$$

$$\cos(x\cos\varphi) = J_0(x)+2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x)\cos 2k\varphi, \tag{6.589}$$

$$\sin(x\cos\varphi) = 2\sum_{k}(-1)^k J_{2k+1}(x)\cos(2k+1)\varphi. \tag{6.590}$$

Bessel's integral. On regarding (6.586), (6.587), (6.589), (6.590) as Fourier expansions, we get

$$\int_0^{\pi}\cos n\varphi\cos(x\sin\varphi)\,d\varphi = \begin{cases}\pi J_n(x) & \text{for } n = 2k, \quad n = 0,\\ 0 & \text{for } n = 2k+1.\end{cases} \tag{6.591}$$

$$\int_0^{\pi}\sin n\varphi\sin(x\sin\varphi)\,d\varphi = \begin{cases}0 & \text{for } n = 0, \quad n, = 2k,\\ \pi J_n(x) & \text{for } n = 2k+1.\end{cases} \tag{6.592}$$

It follows from these formulae that

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\varphi - x\sin\varphi)\,d\varphi \qquad \text{(Bessel)} \tag{6.593}$$

where n is zero or a positive integer.

Addition theorem. Since

$$\sum_{n=-\infty}^{\infty} J_n(u+v)t^n = e^{\frac{1}{2}(u+v)(t-t^{-1})} = e^{\frac{1}{2}u(t-t^{-1})}e^{-\frac{1}{2}v(t-t^{-1})} =$$

$$= \sum_{s=-\infty}^{\infty} J_s(u)t^s \sum_{r=-\infty}^{\infty} J_r(v)t^r,$$

we have

$$J_n(u+v) = \sum_{s=-\infty}^{\infty} J_s(u)J_{n-s}(v), \tag{6.594}$$

or

$$J_n(u+v) = \sum_{s=0}^{n} J_s(u)J_{n-s}(v) +$$

$$+\sum_{s=1}^{\infty}(-1)^s\{J_s(u)J_{n+s}(v)+J_{n+s}(u)J_s(v)\}. \tag{6.595}$$

Similarly,

$$I_n(u+v) = \sum_{s=-\infty}^{\infty} I_s(u)I_{n-s}(v) = \sum_{s=0}^{n} I_s(u)I_{n-s}(v) + $$

$$+\sum_{s=1}^{\infty} \{I_s(u)I_{n+s}(v)+I_{n+s}(u)I_s(v)\}. \qquad (6.596)$$

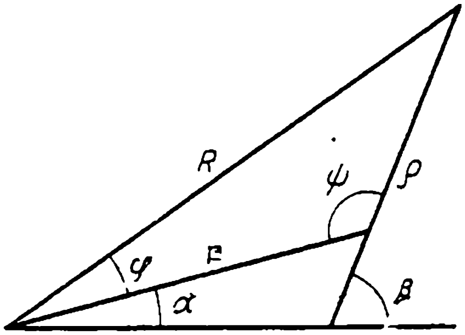


FIG. 16.

The addition theorem in Neumann's form. It follows from Fig. 16 that

$$R = \sqrt{r^2+\varrho^2-2r\varrho\cos\psi},$$

$$R\cos(\varphi+\alpha) = t\cos\alpha+\varrho\cos\beta = r\cos\alpha+\varrho\cos(\pi-\psi+\alpha)=$$

$$= r\cos\alpha+\varrho\cos(\alpha-\psi),$$

whence

$$\sum_n i^n J_n(R)e^{in\varphi}e^{in\alpha} = \sum_m i^m J_m(r)e^{im\alpha}\sum_l (-1)^l i^l J_l(\varrho)e^{il\psi}e^{il\alpha} =$$

$$= \sum_n i^n e^{in\alpha}\sum_{l=-\infty}^{\infty}(-1)^l J_l(\varrho)J_{n-l}(r)e^{-il\psi}, \qquad (6.597)$$

and

$$J_n(R)e^{in\varphi} = \sum_{l=-\infty}^{\infty}(-1)^l J_l(\varrho)J_{n-l}(r)e^{-il\psi}, \qquad (6.598)$$

$$J_n(R)\cos n\varphi = \sum_l (-1)^l J_l(\varrho)J_{n-l}(r)\cos l\psi, \qquad (6.599)$$

$$J_n(R)\sin n\varphi = \sum_l (-1)^{l+1} J_l(\varrho)J_{n-l}(r)\sin l\psi. \qquad (6.600)$$

When $n = 0$,

$$J_0(R) = J_0(R)J_0(\varrho)J_l(r)\cos l\psi \qquad (6.601)$$

When $r = \varrho$, $R = 2\varrho \sin\frac{1}{2}\psi = z \sin\theta$,

$$J_0(z\sin\theta) = J_0^2\left(\frac{z}{2}\right)+2\sum_{l=1}^{\infty} J_l^2\left(\frac{z}{2}\right)\cos 2l\theta\,. \qquad (6.602)$$

Bessel's differential equation, which is satisfied by the Bessel functions $J_n(x)$, is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y = 0. \qquad (6.603)$$

Correspondingly, the $I_n(x)$ satisfy the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+n^2)y = 0. \qquad (6.604)$$

Recurrence relations are:

$$J_{n+1}(x) = \frac{n}{x}J_n(x)-J_n'(x); \qquad (6.605)$$

$$J_{n-1}(x) = \frac{n}{x}J_n(x)+J_n'(x); \qquad (6.606)$$

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x)-J_{n+1}(x)]; \qquad (6.607)$$

$$J_{n+1}(x)+J_{n-1}(x) = \frac{2n}{x}J_n(x); \qquad (6.608)$$

$$\frac{d}{dx}(x^nJ_n) = x^nJ_{n-1}, \quad \frac{d}{dx}(x-nJ_n) = -x^{-n}J_{n+1}. \qquad (6.609)$$

The Bessel functions of the first kind $J_n(x)$, n not an integer,

$$J_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(s+1)\Gamma(n+s+1)}\left(\frac{x}{2}\right)^{n+2s} \qquad (6.610)$$

satisfy the differential equation (6.603).

A second linearly independent solution of this equation is provided by the function $J_{-n}(x)$ (n not an integer), so that the general solution has the form

$$y = C_1 J_n(x) + C_2 J_{-n}(x),$$

where C_1 and C_2 are arbitrary constants. The recurrence formulae (6.605)–(6.609) also hold for $J_n(x)$, where n is not an integer.

Bessel functions of the first kind $J_{n+\frac{1}{2}}(x)$, where n is an integer, are expressible in terms of elementary functions; in particular,

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{\frac{1}{2}} \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\,\Gamma\left(\frac{1}{2}+s+1\right)} \left(\frac{x}{2}\right)^{2s} =$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \sqrt{\frac{2}{\pi}} \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s}}{(2s+1)!} = \sqrt{\frac{2x}{\pi}}\, \frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}} \sin x; \tag{6.611}$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \tag{6.612}$$

Expressions for $J_{n+\frac{1}{2}}(x)$, n any integer, can be obtained by successive applications of the recurrence formulae.

Bessel functions of the second kind $Y_n(x)$ (Weber functions) are expressible in terms of Bessel functions of the first kind with the aid of the equations:

$$Y_n(x) = \lim_{\nu \to n} \frac{J_\nu(x) \cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi} \quad \text{for } n \text{ an integer} \tag{6.613}$$

$$Y_n(x) = \frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi} \quad \text{for } n \text{ not an integer.} \tag{6.614}$$

Like $J_n(x)$, the function $Y_n(x)$ satisfies the differential equation (6.603), the general solution of which has the form, when n is an integer, $y = C_1 J_n(x) + C_2 Y_n(x)$.

The *Macdonald function* is usually taken as the second solution of equation (6.604):

$$K_n(x) = \frac{\pi}{2}\frac{I_{-n}(x)-I_n(x)}{\sin n\pi} \quad \text{for } n \text{ not an integer.} \tag{6.615}$$

The recurrence formulae are:

$$Y_{n+1}(x) = \frac{n}{x}Y_n(x) - Y'_n(x); \tag{6.616}$$

$$Y_{n-1}(x) = \frac{n}{x}Y_n(x) + Y'_n(x); \tag{6.617}$$

$$Y'_n(x) = \frac{1}{2}[Y_{n-1}(x) - Y_{n+1}(x)]; \tag{6.618}$$

$$Y_{n+1}(x) + Y_{n-1}(x) = \frac{2n}{x}Y_n(x); \tag{6.619}$$

$$I_{n+1}(x) - I_{n-1}(x) = -\frac{2n}{x}I_n(x); \tag{6.620}$$

$$K_{n+1}(x) - K_{n-1}(x) = \frac{2n}{x}K_n(x). \tag{6.621}$$

Relationships between Bessel functions of the first and second kinds are:

$$J_n(x)Y_{n+1}(x) - Y_n(x)J_{n+1}(x) = -\frac{2}{\pi x}; \tag{6.622}$$

$$I_n(x)K_{n+1}(x) + K_n(x)I_{n+1}(x) = \frac{1}{x}. \tag{6.623}$$

Representations as series are:

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{2}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \dots; \tag{6.624}$$

$$J_1(x) = \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 - \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^7 + \dots; \tag{6.625}$$

$$Y_0(x) = \frac{2}{\pi}\left(\mathbf{C} + \ln\frac{x}{2}\right)J_0(x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\sum_{k=1}^{n}\frac{1}{k}; \tag{6.626}$$

$$Y_1(x) = \frac{2}{\pi}\left(\mathbf{C}+\ln\frac{x}{2}\right)J_1(x) - \frac{2}{\pi x} - \\ -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+1)!}\left(\frac{x}{2}\right)^{2n+1}\left(2\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{n+1}\right); \tag{6.627}$$

$$I_0(x) = 1+\left(\frac{x}{2}\right)^2+\frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4+\frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6+\dots; \tag{6.628}$$

$$I_1(x) = \frac{x}{2}+\frac{1}{2!}\left(\frac{x}{2}\right)^3+\frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5+\frac{1}{3!\,4!}\left(\frac{x}{2}\right)^7+\dots; \tag{6.629}$$

$$K_0(x) = -\left(\mathbf{C}+\ln\frac{x}{2}\right)I_0(x)+\sum_{n=1}^{\infty}\frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\sum_{k=1}^{n}\frac{1}{k}; \tag{6.630}$$

$$K_1(x) = \left(\mathbf{C}+\ln\frac{x}{2}\right)I_1(x)+\frac{1}{x} - \\ -\sum_{n=0}^{\infty}\frac{1}{n!\,(n+1)!}\left(\frac{x}{2}\right)^{2n+1}\left(\sum_{k=1}^{n+1}\frac{1}{k}-\frac{1}{2n+2}\right); \tag{6.631}$$

where **C** is Euler's constant (see § 1, sec. 1, 3°).

Asymptotic formulae (for large values of x) are:

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}}\left[P_0(x)\sin\left(x+\frac{\pi}{4}\right)+Q_0(x)\cos\left(x+\frac{\pi}{4}\right)\right]; \tag{6.632}$$

$$J_1(x) \approx \sqrt{\frac{2}{\pi x}}\left[P_1(x)\sin\left(x-\frac{\pi}{4}\right)+Q_1(x)\cos\left(x-\frac{\pi}{4}\right)\right]; \tag{6.633}$$

$$Y_0(x) \approx \sqrt{\frac{2}{\pi x}}\left[-P_0(x)\cos\left(x+\frac{\pi}{4}\right)+Q_0(x)\sin\left(x+\frac{\pi}{4}\right)\right]; \tag{6.634}$$

$$Y_1(x) \approx \sqrt{\frac{2}{\pi x}}\left[-P_1(x)\cos\left(x-\frac{\pi}{4}\right)+Q_1(x)\sin\left(x-\frac{\pi}{4}\right)\right]; \tag{6.635}$$

where

$$P_0(x) = 1-\frac{1^2.3^2}{2!\,(8x)^2}+\frac{1^2.3^2.5^2.7^2}{4!\,(8x)^4}-\dots;$$

$$Q_0(x) = -\frac{1}{1!\,8x}+\frac{1^2.3^2.5^2}{3!\,(8x)^3}-\dots;$$

$$P_1(x) = 1+\frac{1^2.3.5}{2!\,(8x)^2}-\frac{1^2.3^2.5^2.7.9}{4!\,(8x)^4}+\dots;$$

$$Q_1(x) = \frac{1.3}{1!\,8x}-\frac{1^2.3^2.5.7}{3!\,(8x)^3}+\dots;$$

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}\left(1+\frac{1^2}{8x}+\frac{1^2.3^2}{2!\,(8x)^2}+\frac{1^2.3^2.5^2}{3!(8x)^3}+\ldots\right); \qquad (6.636)$$

$$I_1(x) \approx \frac{e^x}{\sqrt{2\pi x}}\left(1-\frac{1.3}{8x}-\frac{1^2.3.5}{2!\,(8x)^2}-\frac{1^2.3^2.5.7}{3!\,(8x)^3}-\ldots\right); \qquad (6.637)$$

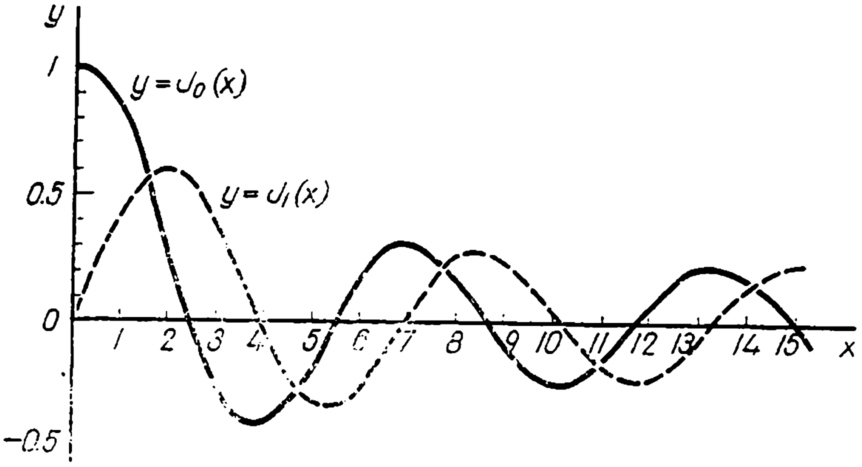


FIG. 17. Bessel functions of the first kind $J_0(x)$ and $J_1(x)$.

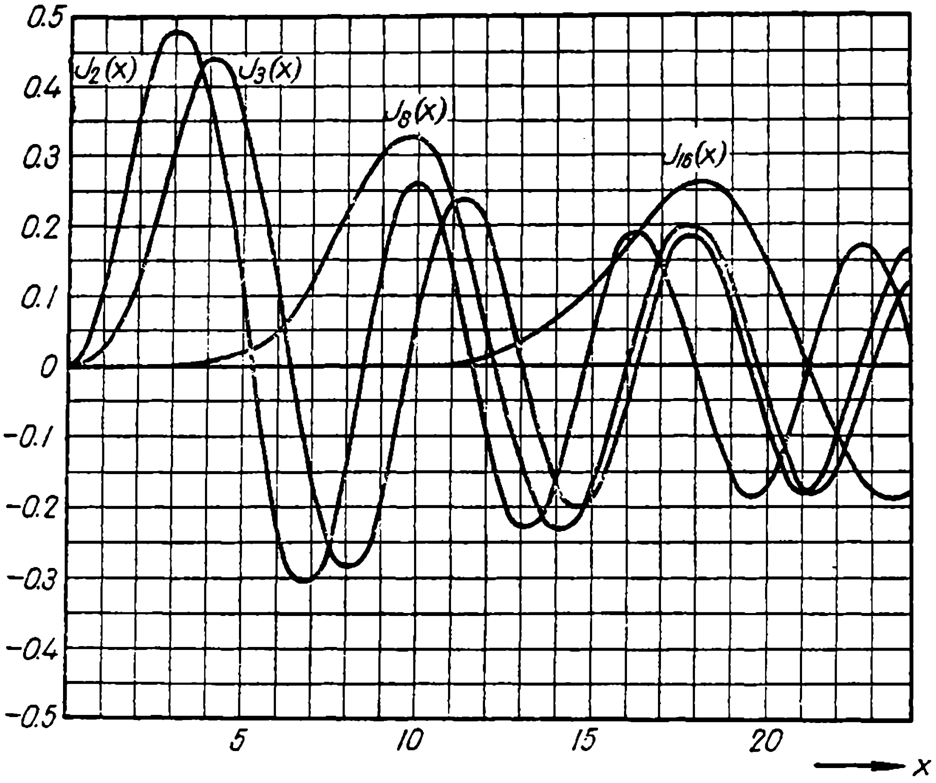


FIG. 18. Bessel functions of the first kind $J_2(x)$, $J_3(x)$, $J_8(x)$, $J_{16}(x)$.

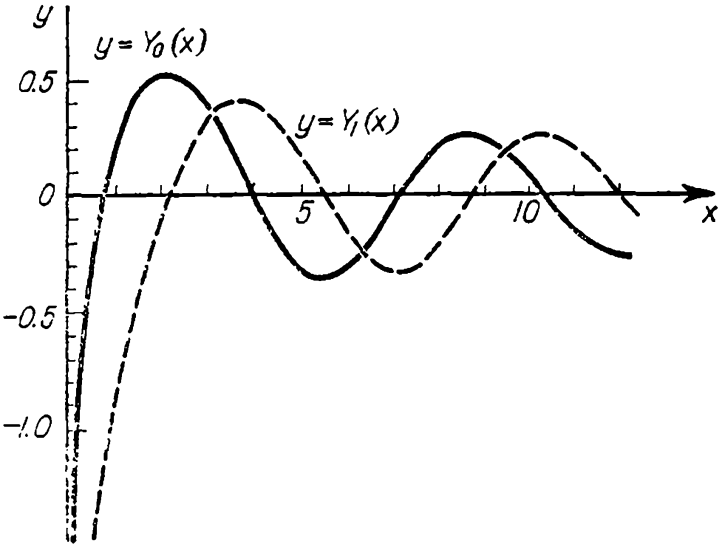
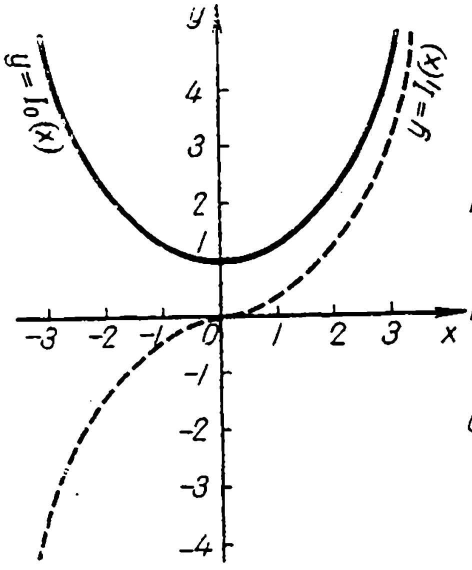


FIG. 19. Bessel functions of the second kind $Y_0(x)$ and $Y_1(x)$.



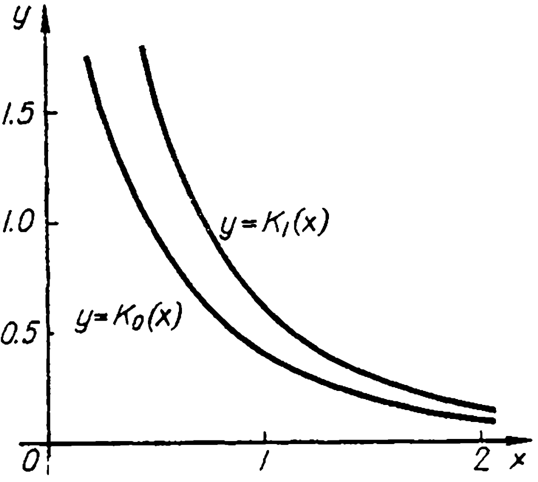


FIG. 20. Bessel functions of the first kind $I_0(x)$ and $I_1(x)$.

FIG. 21. Bessel functions of the second kind $K_0(x)$ and $K_1(x)$.

$$K_0(x) \approx e^{-x}\sqrt{\frac{\pi}{2x}}\left(1-\frac{1^2}{8x}+\frac{1^2.3^2}{2!\,(8x)^3}-\frac{1^2.3^2.5^3}{3!\,(8x)^3}+\dots\right); \tag{6.638}$$

$$K_1(x) \approx e^{-x}\sqrt{\frac{\pi}{2x}}\left(1+\frac{1.3}{8x}-\frac{1^2.3.5}{2!\,(8x)^2}+\frac{1^2.3^2.5.7}{3!\,(8x)^3}-\dots\right). \tag{6.639}$$

Graphs of Bessel functions are shown in Fig. 17, 18, 19, 20 and 21.

NOMENCLATURE

$(a\ b)$ – The interval $a < x < b$

$[-\infty, a)$ – The infinite interval $x < a$

$[b, +\infty)$ – The infinite interval $x > b$

$[a, b]$ – The segment $a \leq x \leq b$

$(a, b]$ – The semi-interval $a < x \leq b$

$[a, b)$ – The semi-interval $a \leq x < b$

$(-\infty, a]$ – The infinite semi-interval $x \leq a$

$[b, +\infty)$ – The infinite semi-interval $x \geq b$

$\{x_n\}$ – The set of elements x_n

$x \in X$ – The element x belongs to the set X

$x \bar{\in} X$ or $x \notin X$ – The element x does not belong to the set X

$X \subset Y$ – The set X is a subset of the set Y

$X \bar{\subset} Y$ or $X \not\subset Y$ – The set X is not a subset of the set Y

$A \cup B$ or $A+B$ – The union (sum) of sets A and B

$A \cap B$, or $A \times B$, or $A \cdot B$ or AB – The intersection (product) of sets A and B

$B \setminus A$ or $B-A$ – The complement of the set A with respect to the set B

$\underset{x \in X}{x}$ – The strict upper bound of the set X

$\underset{x \in X}{\sup} f(x)$ – The strict upper bound of the function f on the set X

$\underset{x \in X}{\inf}\, x$ – The strict lower bound of the set X

$\underset{x \in X}{\inf} f(x)$ – The strict lower bound of the function f on the set X

$\underset{n}{\max}\{a_1, a_2, \ldots, a_n\}$ – The greatest of the numbers $a_1, a_2, \ldots, a_n$

$\underset{n}{\min}\{a_1, a_2, \ldots, a_n\}$ – The least of the numbers $a_1, a_2, \ldots, a_n$

$\{a_n\} = \{a_1, a_2, \ldots, a_n, \ldots\}$ – The sequence with the general term a_n

$\{a_{nm}\}$ – A double sequence

$o(\alpha_n)$ – α_n is an infinitesimal of lower order with respect to β_n if $\beta_n = O(\alpha_n)$

$O(\alpha_n)$ – α_n has a rate of decrease not faster than β_n

$o(x)$, $O(x)$ – Orders of the function x

$\lim\limits_{x \to x_0,\, x \in X} f(x)$ – The limit of the function $f(x)$ as $x \to x_0$, $x \in X$

E_1 – One-dimensional coordinate space (numerical axis)

E_n – n-dimensional coordinate space

$E_k + E_{n-k}$ – The direct sum of manifolds

$\varrho(X, Y)$ – A distance

$\theta(0, 0, \ldots, 0)$ – The origin of coordinates
$f(X) = f(x_1, x_2, \ldots, x_n)$ – A linear function, or a function of a vector (point)
$\|X\|$ – The norm of the vector X
$\Gamma_{x_1 x_2 \ldots x_n}$ – Gram's determinant for vectors
L_n – n-dimensional linear system
$X \perp E_k$ – The vector X of E_n is orthogonal to E_k
$\mathrm{pr}_{E_k} X$ – The projection of the vector X on to E_k
$\mathrm{pr}_{U_0} X$ – The projection of the vector X on to the vector U_0
$\lim\limits_{X \to A (X \in M)} f(X)$ – The limit of the function $f(X)$, when $X \in M$ tends to A
Q_e – The exterior domain
Q_i – The interior domain
$F = (f_1, f_2, \ldots, f_m)$ – An operator
$Y = F(X)$ – The vector form of writing an operator
$y_i = f_i(X) = f_i(x_1, x_2, \ldots, x_n)$ – The coordinate form of writing an operator
$y_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$ $(i = 1, 2, \ldots, n)$ – The coordinate form of writing a linear operator
$E_{n\varphi}$ – The space with norm $\varphi(X)$
$E_{n\psi} = E^*_{n\varphi}$ – The space $E_{n\psi}$ is conjugate to the space $E_{n\varphi}$
$P_m = p_1 p_2 \ldots p_m$ – The mth partial product
$\{P_m\}$ – The sequence of partial products
$\pi_m = p_{m+1}p_{m+2} \ldots = \prod\limits_{n=m+1}^{\infty} p_n$ – The remainder product
$\sum\limits_{k,l=0}^{\infty} a_{kl}$ – A double series
R – The radius of convergence of a power series
$r_n(x)$ – The remainder term of the Taylor series
$F(x) \sim \sum\limits_{k=0}^{\infty} A_k/x^k$ – The asymptotic expansion of the function $F(x)$
$(f, g) = \int_a^b f(x)g(x)\, d\sigma(x)$ – The scalar (inner) product of the functions $f(x)$ and $g(x)$
$\|f\|$ – The norm of a function
$\|f - g\| = \sqrt{\int_a^b (f-g)^2\, d\sigma(x)}$ – The root square deviation of functions f and g
$r_n f(x)$ – The nth-order inverse derivative of $f(x)$
$b_0 + \frac{a_1}{b_1} \, \frac{a_2}{+b_2} + \ldots \frac{a_n}{+b_n} + \ldots$ – A non-terminating continued fraction
P_n/Q_n – The nth convergent of the fraction
$K, \tilde{K}$ – The ordinary and singular values of the same continued fraction
C – The Euler–Mascheroni constant
G – Catalan's constant

$n!$ – Factorial n
$n!!$ – Double factorial n
δ_{nm} or δ_n^m Kronecker delta.
$\binom{n}{m}$ or C_n^m – The binomial coefficients
$P_n(x)$, $\overline{P}_n(x)$, $\hat{P}_n(x)$ – Arbitrary orthogonal polynomials
B_n – Bernoulli numbers
$B_n(x)$ – Bernoulli polynomials
$\varphi_n(x) = B_n(x) - B_n$
$B_n(f)$ – Bernshtein polynomials
$L_n^{\alpha}(x)$, $\overline{L}_n^{\alpha}(x)$, $\hat{L}_n^{\alpha}(x)$ – Laguerre polynomials
$L_n(x)$, $\overline{L}_n(x)$, $\hat{L}_n(x)$, l_n – Legendre polynomials
$T_n(x)$, $\overline{T}_n(x)$, $\hat{T}_n(x)$ – Chebyshev polynomials of the first kind
$\overline{U}_n(x)$, $\hat{U}_n(x)$, $U_n(x)$ – Chebyshev polynomials of the second kind
$P_{k,N}(x)$ – Chebyshev polynomials with respect to a system of points
E_n – Euler numbers
$E_n(x)$ – Euler polynomials
$H_n(x)$, $\overline{H}_n(x)$, $\hat{H}_n(x)$ – Hermite polynomials
$J_n^{(\lambda,\mu)}(x)$, $\overline{J}_n^{(\lambda,\mu)}(x)$, $\hat{J}_n^{(\lambda,\mu)}(x)$, $j_n^{(\lambda,\mu)}(x)$ – Jacobi polynomials
$|x|$ – The absolute value of x
sign x – The sign of x
$[x]$ or $E(x)$ – The integral part of x
$\{x\}$ – The fractional part of x
(x) – The distance to the nearest integer
$1(x)$ – Heaviside's unit function
$\delta(x)$ – The delta function
$B(\alpha, \beta)$ – Euler's beta function
$\Gamma(\alpha)$ – Euler's gamma function
$\mathrm{Ei}(x)$ – The integral exponential function
$\overline{\mathrm{Ei}}(x)$ – The real part of $\mathrm{Ei}(x)$
li x – The integral logarithm
si (x) or Si x – The integral sine
ci x or Ci x – The integral cosine
erf x, $\varphi(x)$, $\varphi_B(x)$, erfc x, $L(x)$ – The error function.
$F(k, \varphi)$ – The elliptic integral of the first kind
$E(k, \varphi)$ – The elliptic integral of the second kind
$\Pi(k, \lambda, \varphi)$ – The elliptic integral of the third kind
$D(k, \varphi)$ – $[F(k, \varphi) - E(k, \varphi)]/k^2$
K – The complete elliptic integral of the first kind
E – The complete elliptic integral of the second kind
$D = (K-E)/k^2$
$\psi(\alpha)$, $\Psi(\alpha)$ – The logarithmic derivative of the gamma function (the psi functions)
$S(x)$, $S^*(x)$ – The Fresnel sine integral
$C(x)$, $C^*(x)$ – The Fresnel cosine integral

$J_n(x)$ – Bessel function of the first kind
$Y_n(x)$ – Bessel function of the second kind
$I_n(x)$ – Bessel function of an imaginary argument
$K_n(x)$ – Macdonald function

REFERENCES

CHAPTER I

1. I. N. BRONSHTEIN and K. A. SEMENDYAYEV, *A Guide-Book to Mathematics for Technologists and Engineers (Spravochnik po matematike dlya inzhenerov i uchashchikhsya vtuzov)*, Fizmatgiz, Moscow (1959). English translation by Pergamon Press,(1963).
2. CH. DE LA VALLÉE-POUSSIN, *Cours d'Analyse Infinitésimale*, Gauthier-Villars, Paris-Louvain (1937–47).
3. E. GOURSAT, *Cours d'Analyse Mathematique*, 5th ed., Gauthier-Villars, Paris (1942).
4. A. YA. DUBOVITSKII, Axiomatic construction of the real numbers, *Matematicheskoye prosveshchenie*, No. 2 (1957).
5. M. A. KARTSEV, *The Arithmetical Principles of Electronic Digital Computers (Arifmeticheskiye ustroistva elektronnykh tsifrovykh mashin)*, Fizmatgiz, Moscow (1958).
6. A. N. KOLMOGOROV, A note on the foundations of the theory of real numbers, *Matematicheskoye prosveshchenie*, No. 2 (1957).
7. A. N. KOLMOGOROV, The basis of the method of least squares, *Usp. Mat. Nauk* 1, No. 1 (1946).
8 M. G. KRASNOSEL'SKII, *Convex Functions and Orlicz Spaces (Vypuklyye funktsii i prostranstva Orlicha)*, Fizmatgiz, Moscow (1958).
9. N. N. LUZIN, *The Theory of Functions of a Real Variable (Teoriya funktsii deistvitel'nogo peremennogo)*, Uchpedgiz, Moscow-Leningrad (1940).
10. V. NEMYTSKII, M. SLUDSKAYA and A. CHERKASOV, *A Course of Mathematical Analysis (Kurs matematicheskogo analiza)*, Vol. I and II, Gostekhizdat, Moscow (1957).
11. G. POLYA and G. SZEGO, *Aufgaben und Lehrsatze aus der Analysis*, 2 volumes. Springer, Berlin (1925); reprinted, Dover, N.Y. (1945).
12. V. I. SMIRNOV, *A Course of Higher Mathematics (Kurs vysshei matematiki)*, Vol. I, Gostekhizdat, Moscow-Leningrad (1948). English translation by Pergamon Press, (1964).
13. E. T. WHITTAKER and G. N. WATSON, *A Course in Modern Analysis*, 4th ed., Cambridge Univ. Press (1940).
14. G. M. FIKHTENGOL'TS, *A Course of Differential and Integral Calculus (Kurs differentsial'nogo i integral'nogo ischisleniya)*, 3 volumes, Gostekhizdat, Moscow–Leningrad (1951, 1951, 1949).

15. A. YA. KHINCHIN, *A Short Course of Mathematical Analysis (Kratkii kurs matematicheskogo analiza)*, Fizmatgiz, Moscow (1959).
16. A. YA. KHINCHIN, The elementary linear continuum. *Usp. Mat. Nauk.* **4**, No. 2 (1949).

CHAPTER II

1. A. D. ALEKSANDROV, *The Interior Geometry of Convex Surfaces (Vnutrennyaya geometriya vypuklykh poverkhnostei)*, Gostekhizdat, Moscow-Leningrad (1948).
2. A. D. ALEKSANDROV, *Convex Polyhedra (Vypuklyye mnogogranniki)*, Moscow-Leningrad (1950).
3. D. HILBERT and S. COHN-VOSSEN, *Anschauliche Geometrie*, Springer, Berlin (1932).
4. M. A. LAVRENT'EV and L. A. LYUSTERNIK, *Foundations of the Calculus of Variations (Osnovy variatsionnogo ischisleniya)*, Vol. I, Part I, United Scientific and Technical Press, Moscow-Leningrad (1935).
5. L. A. LYUSTERNIK, *Convex Figures and Polyhedra (Vypuklyye figury i mnogogranniki)*, Gostekhizdat, Moscow (1956).
6. L. A. LYUSTERNIK, Applications of the Brunno–Minkovskii inequality to extremal problems, *Usp. Mat. Nauk*, No II (1936).
7. L. A. LYUSTERNIK, The Brunno–Minkovskii inequality for arbitrary measurable sets, *Dokl. Akad. Nauk*, **3**, pp. 55–8 (1935).
8. L. V. KANTOROVICH and V. V. NOVOZHILOV, (Editors), *Linear Inequalities and Associated Problems (Lineinyye neravenstva i smezhnyye voprosy)*; a collection of translations (Sb. perevodov), Foreign Literature Publishing House, Moscow (1959).
9. G. MINKOVSKII, General theorems on convex polyhedra, *Usp. Mat. Nauk*, No. II (1936).
10. V. I. SMIRNOV, *A Course of Higher Mathematics (Kurs vysshei matematiki)* Vol. I, Gostekhizdat, Moscow-Leningrad (1948) English translation by Pergamon Press, (1964).
11. G. M. FIKHTENGOL'TS, *A Course of differential and Integral Calculus (Kurs differentsial'nogo i integralnogo ischisleniya)*, Vol. I. Gostekhizdat, Moscow-Leningrad (1951).
12. E. HELLY, A set of convex bodies with common points, *Usp. Mat. Nauk*, No. II (1936).
13. G. E. SHILOV, *An Introduction to the Theory of Linear Spaces (Vvedenie v teoriyu lineinykh prostranstv)*. Gostekhizdat, Moscow (1952).
14. D. O. SHKLYARSKII, Conditionally convergent vector series, *Usp. Mat. Nauk*, No. X, pp. 51–9 (1944).
15. O. SCHREIER and E. SPERNER, *Einführung in die Analytische Geometrie und Algebra*, 2 volumes, Vandenhoeck and Ruprecht, Göttingen (1948–51).
16. I. M. YAGLOM and V. G. BOLTYANSKII, *Convex Figures (Vypuklyye figury)*, Gostekhizdat, Moscow-Leningrad (1951).

CHAPTER III

1. YA. I. ALIKHASHKIN, A method of computing the debit for the head flow to an imperfect bore-hole, *Vychislit. matem.* No. 1 (1957).
2. E. GOURSAT, *Cours d'Analyse Mathematique*, Vol. I, Gauthier-Villars, Paris. (1942).
3. L. V. KANTOROVICH and V. I. KRYLOV, *Approximation Methods of Higher Analysis (Priblizhennyye metody vysshego analiza)*, 4th ed., Gostekhizdat, Moscow--Leningrad (1952).
4. I. I. PRIVALOV, *Fourier Series (Ryady Furi'e)*, United Scientific and Technica Press, Moscow-Leningrad (1934).
5. V. I. ROMANOVSKII, *An Introduction to Analysis (Vvedeniye v analiz)*, Gosuch pedizdat, Tashkent, UzSSR (1939).
6. I. M. RYZHIK and I. S. GRADSHTEIN, *Tables of Integrals, Sums, Series and Products (Tablitsy integralov, summ, ryadov i proizvedenii)*, Gostekhizdat, Moscow-Leningrad (1951).
7. G. S. SALEKHOV, *The Computation of Series (Vychisleniye ryadov)*, Gostekhizdat, Moscow (1955).
8. G. S. SALEKHOV, The application of Laplace transforms to the summation of series expanded in special functions, *Uch. zap. Kazansk.* ped. in-ta, No. 10 (1955).
9. G. P. TOLSTOV, *Fourier Series (Ryady Furi'e)*, Gostekhizdat, Moscow-Leningrad (1951).
10. G. M. FIKHTENGOL'TS, *A Course of Differential and Integral Calculus (Kurs differentsial'nogo i integral'nogo ischisleniya)*, Vol. II and III ,Gostekhizdat, Moscow-Leningrad (1951, 1949).
11. G. H. HARDY, *Divergent Series*, Clarendon Press, Oxford (1949).
12. K. KNOPP, *Theorie und Anwendungen der unendlichen Reihen*, Berlin (1924).

CHAPTER IV

1. YA. L. GERONIMUS, *The Theory of Orthogonal Polynomials (Teoriya orthogonal'nykh mnogochlenov)*, Gostekhizdat, Moscow-Leningrad (1950).
2. D. JACKSON, Fourier Series and Orthogonal Polynomials, *Carus Monograph of Math. Assoc. of Am.*, Oberlin, Ohio (1941).
3. L. N. KARMAZINA, Tables of Jacobi Polynomials, (Tablitsy mnogochlenov Jakobi), *Izd. Akad. Nauk SSSR*, Moscow (1954).
4. S. KACZMARZ and H. STEINHAUS, *Theorie der Orthogonalreihen*, Chelsea, N.Y. (1951).
5. M. A. LAVRENT'EV and B. V. SHABAT, *Methods of the Theory of Functions of a Complex Variable (Metody teorii funktsii kompleksnogo peremennogo)*, Fizmatgiz, Moscow (1958).
6. L. A. LYUSTERNIK, On the computation of the values of functions of one variable, *Matematicheskoye prosveshcheniye*, No. 3 and No. 4 (1958).

7. I. P. NATANSON, *The Constructive Theory of Functions (Konstruktivnaya teoriya funktsii)*, Gostekhizdat, Moscow-Leningrad (1947).
8. I. M. STESIN, The conversion of orthogonal expansions into continued fractions, *Vychisl. matem.*, No. 1 (1957).
9. A. S. HOUSEHOLDER, *Principles of Numerical Analysis*, McGraw-Hill, N.Y. (1953).
10. P. L. CHEBYSHEV, Collected papers, (Sobraniye sochinenii), Vol. I, *Izd. Akad. Nauk SSSR* Moscow (1944).
11. O. PERRON, *Die Lehre von der Ketterbrücken*, Leipzig und Berlin (1928).

CHAPTER V

1. I. V. ARNOLD, *The Theory of Numbers (Teoriya chisel)*, Uchpedgiz, Moscow (1939).
2. J. L. F. BERTRAND, *Traité d'Algebre*, Hachette, Paris (1866–7).
3. I. M. VINOGRADOV, *Foundations of the Theory of Numbers (Osnovy teorii chisel)*, Gostekhizdat, Moscow-Leningrad (1949).
4. N. N. VOROB'EV, *Fibonacci Numbers (Chisla Fibonachchi)*, Gostekhizdat, Moscow (1951). English translation by Pergamon Press (1961).
5. A. CAUCHY, *Cours d'Analyse Algébraique*, Leipzig (1864).
6. N. I. LOBACHEVSKII, *Algebra or the Evaluation of Finites (Algebra ili vychisleniye konechnykh)*, Vol. IV of the complete works, Gostekhizdat, Moscow-Leningrad (1948).
7. N. N. MARAKUYEV, *Elementary Algebra (Elementarnaya algebra)*, Vol. I, Theory, (Teoriya), Moscow (1903).
8. A. A. MARKOV, *Lectures on Continued Fractions (Lektsii o nepreryvnykh drobyakh)*. See Collected works on *The Theory of Continued Fractions and the Theory of Functions that Least Deviate from Zero (Izbrannye trudy po teorii nepreryvnykh drobei i teorii funktsii, naimenee uklonyayushchikhsya ot nulya)* Gostekhizdat, Moscow-Leningrad (1948).
9. A. MESHKOV, *A Course of Higher Algebra (Kurs vysshei algebry)*, Part I, SPb (1862).
10. M. V. OSTROGRADSKII, *Lectures on Algebraic and Transcendental Analysis (Lektsii algebraicheskogo i transcendentnogo analiza)*, First Year, SPb (1837).
11. P. ROSHCHIN, *Notes on the differential and Integral Calculus (Zapiski po differential'nomu i integral'nomu ischisleniyam)*, Part I, SPb (1888).
12. B. I. SEGAL, Continued fractions, *Matematicheskoye prosveshcheniye*, No. 7 (1936).
13. T. I. STIELTJES, *Recherches sur les fractions continues oeuvres*, **2**, (1894).
14. A. K. SUSHKEVICH, The Theory of Numbers (Teoriya chisel), Khar'kov, *Izd. Khar'k. un-ta* (1954).
15. K. FERBER, *Arithmetic (Arifmetika)* (1914).
16. A. YA. KHINCHIN, *Continued Fractions (Tsepnyye drobi)*, Gostekhizdat, Moscow-Leningrad (1949).

17. A. Ya. Khinchin, Elements of the theory of numbers (Elementy teorii chisel) *Encyclopedia of Elementary Mathematics*, Book I (*Entsiklopediya elementarnoi matematiki*, Kniga pervaya), Gostekhizdat, Moscow-Leningrad (1951).
18. A. N. Khovanskii, *The Application of Continued Fractions and Generalizations of them to Problems of Approximate Analysis (Prilozheniye tsepnykh drobei i ikh obobshchenii k voprosam priblizhennogo analiza)*, Gostekhizdat, Moscow (1956).
19. A. N. Khovanskii, Euler's works on the theory of continued fractions, *Istor. matem. issled.* No. 10, 305–326 (1957).
20. N. G. Chebotarev, *The Theory of Continued Fractions (Teoriyanepreryvnykh drobei)*, Kazan' (1938).
21. L. D. Eskin, A note on Euler's algorithm for the extraction of roots (Ob algoritme Eilera dlya izvlecheniya kornei), *Uchen. zap. Kazansk. un-ta* 115, book 14, pp. 139–43 (1955).
22. L. D. Eskin, The problem of the nature of the convergence of Euler's algorithm for the extraction of roots, *Iz. vyssh. uchebn. zaved.*, Matem. 4 (5), pp. 275–8 (1958).

Chapter VI

1. Ch. de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, Gauthier-Villars Paris-Louvain (1937–47).
2. I. M. Gel'fand and G. E. Shilov, *Generalized Functions and Operations on them (Obobshchennyye funktsii i deistviya nad nimi)*, Fizmatgiz, Moscow (1958).
3. A. O. Gel'fond, *The Calculus of Finite Differences (Ischisleniye konechnykh raznostei)*, Gostekhizdat, Moscow-Leningrad (1952).
4. I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Sums, Series and Products (Tablitsy integralov, summ, ryadov i proizvedenii)*, Gostekhizdat, Moscow-Leningrad (1951).
5. D. Ivanenko and A. Sokolov, *The Classical Theory of Fields (Klassicheskaya teoriya polya)*, Gostekhizdat, Moscow-Leningrad (1949).
6. V. A. Kudriavtsev, *The Summation of Powers of Natural Numbers and Bernoulli Numbers (Summirovanie stepenei chisel naturalnogo ryada i chisla Bernoulli)*, United Scientific and Technical Press, Moscow-Leningrad (1938).
7. E. Landau, *Einführung in die Differentialrechnung und Integralrechnung*, Noordhoff, Groningen (1934). English translation: Chelsea, N.Y. (1951).
8. E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, 4th ed., Cambridge Univ. Press (1940).
9. G. M. Fikhtengol'ts, *A Course of Differential and Integral Calculus (Kurs differentsial'nogo i integral'nogo ischisleniya)*, Vol. I, II, III, Gostekhizdat, Moscow-Leningrad (1951, 1951, 1949).
10. G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford (1949).
11. E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung*, Part I, Leipzig (1904).

12. E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung*, Part II, Leipzig (1904).
13. Ya. N. Spielrein, *Tables of Special Functions (Tablitsy spetsial'nykh funktsii)*, Parts I and II, GTTI, Moscow-Leningrad (1933).
14. E. Jahnke and F. Emde, *Funktionentafeln*, Chelsea, N.Y. (1951).
15. H. Bateman, *Higher Transcendental Functions*, N.Y. (1953).
16. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin (1924).

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